Lecture One

Topolgy *

1 Logic and Set Theory

Definition 1. *A set is a collection of definite and distinguishable objects, these objects are called elements.*

Example 1. $A = \{a, e, i, o, 4\}$ *is a set of 5 elements and* $a \in A$.

Example 2. $B = \{x : x\, is\, an\, integer\, and\, x > 0\}$ *is a set which is written in builder notation. Note that* $\pi \notin B$ *while* $6 \in B$.

1.1 Subsets

A set A is a subset of B written as $A \subseteq B$ if and only if $x \in A \Rightarrow x \in B$ and is read as A is subset of B or A is contained in B or B contains A.

1.2 Equality of Sets

Definition 2. *Any two sets* A *and* B *are said to be equal if and only if* $A \subseteq B$ *and* $B \subseteq A$.

1.3 Set Operations

Let A and B be two sets. Then the following operations are defined.

- $A \cup B = \{x : x \in A\, or\, x \in B\}$.
- $A \cap B = \{x : x \in A\, and\, x \in B\}$.
- $A \setminus B = \{x : x \in A\, and\, x \notin B\}$.
- $A^c = \{x : x \in U\, and\, x \notin A\}$, where U is the universal set.

Note that

- $\phi^c = U$.
- $U^c = \phi$.

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- $A \cup A^c = U$.
- $A \cap A^c = \phi$.

1.4 Disjoint Sets

Two sets A and B are called disjoint if $A \cap B = \phi$.

1.5 Laws of the Algebra of Sets

Let A, B, and C be sets with U as the universal set. Then

- $A \cup A = A$.
- $A \cap A = A$.
- $A \cap (B \cap C) = (A \cap B) \cap C$.
- $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \cup B = B \cup A$.
- $A \cap B = B \cap A$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- $A \cup U = U$.
- $A \cap U = A$.
- $A \cap \phi = \phi$.
- $A \cup \phi = A$.
- $(A^c)^c = A$.

1.6 Product of the Sets

The product set of A and B, written as $A \times B$ consists of all the ordered pairs (a, b) where $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Example 3. *Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then*

$$A \times B = \{(1, a)(1, b), (2, a)(2, b)(3, a)(3, b)\}.$$

Remark 1. *The concept of product of sets can be extended to any finite number of sets in a natural way. The product of the sets $A_1, \ldots, A_n$ denoted by $\prod_{i=1}^{n} A_i$ consists of all n tuples $(a_1, \ldots, a_n)$, where $a_i \in A_i$ for each i.*

1.7 Indexed Sets

Let $\triangle$ be a nonempty set such that $\forall \alpha \in \triangle$, there is a particular set X_α. Then the set $\{X_\alpha : \alpha \in \triangle\}$ is called a family set and $\triangle$ is called the indexing set.

Example 4. *Let* $\triangle = \{0, 1, 2, 3, 4\}$*, and*

$$X_\alpha = \{2\alpha + 4, 18, 12 - 2\alpha\} \ \textit{such that}\, \alpha \in \triangle.$$

e.g., $X_0 = \{4, 18, 12\}$, $X_1 = \{6, 18, 10\}$, $X_2 = \{8, 18\}$, $X_3 = \{10, 18, 6\}$, $X_4 = \{12, 18, 4\}$.

Example 5. *Let* $\triangle = \mathbb{N}$*, and*

$$X_\alpha = \{\alpha, \alpha + 1, \alpha^2, \} \ \textit{such that}\, \alpha \in \triangle.$$

e.g., $X_1 = \{1, 2\}$, $X_2 = \{2, 3, 4\}, \ldots.$

Note that

- $\bigcup\limits_{\alpha \in \triangle} X_\alpha = \{x : x \in X_\alpha \, \textit{for some}\, \alpha \in \triangle\}$.
- $\bigcap\limits_{\alpha \in \triangle} X_\alpha = \{x : x \in X_\alpha \, \textit{for all}\, \alpha \in \triangle\}$.
- $(\bigcup\limits_{\alpha \in \triangle} X_\alpha)^c = \bigcap\limits_{\alpha \in \triangle} X_\alpha^c$.
- $(\bigcap\limits_{\alpha \in \triangle} X_\alpha)^c = \bigcup\limits_{\alpha \in \triangle} X_\alpha^c$.

Example 6. *In Example 4,* $\bigcup\limits_{\alpha \in \triangle} X_\alpha = \{4, 18, 12, 6, 10, 8\}$ *and* $\bigcap\limits_{\alpha \in \triangle} X_\alpha = \{18\}$.

1.8 Power Set

Definition 3. *Let A be nonempty set. Then $\mathcal{P}(A) = \{G : G \subseteq A\}$ is called the power set of A.*

Example 7. *Let $A = \{a, b, c\}$. The power set of A is*

$$\mathcal{P}(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \phi\}.$$

Remark 2. *If A has n elements, then $\mathcal{P}(A)$ have 2^n elements.*

Lecture Two

1.9 Functions

Definition 4. *A function*

$$f : A \to B,$$

is a relation such that

- *Domain* $(f) = A$.
- *If* $(x, y) \in f$ *and* $(x, z) \in f$, *then* $y = z$.

Definition 5. *A function*

$$f : A \to B,$$

is said to be one-one or injective if the distinct elements in A *have distinct images, i.e.,*

$$f(a) = f(b) \Rightarrow a = b$$

Definition 6. *A function*

$$f : A \to B,$$

is said to be onto or surjective if

$$\forall b \in B, \exists a \in A \, such \, that \, f(a) = b.$$

Remark 3. *If* f *is one-one and onto, then* f *is called bijective and* f^{-1} *exists.*

Remark 4.

- *If* $f : X \to Y$ *be any function, then* $f(X) \subseteq Y$, *If* f *is onto, then* $f(X) = Y$.
- $f^{-1}(Y) = X$ *is always true.*

Theorem 1. *Let* $f : A \to B,$, $g : B \to C$, *be two functions.* $g \circ f : A \to C$ *is a function and* $g \circ f(a) = g(f(a))$ *for every* $a \in A$.

1.10 Equivalent Sets

Definition 7. *A set* A *is said to be equivalent to a set* B *and we write it as* $A \sim B$ *(read as* A *is equivalent to* B *) if there exists a function*

$$f : A \to B,$$

which is one-one and onto.

Definition 8. *For each natural number* k, *Let* $N_k = \{1, 2, \ldots, k\}$. *A set* A *is finite if and only if* $A = \phi$ *or* $A \sim N_k$.

Example 8. *The set* $s = \{1, \frac{1}{2}, c, 99\}$ *is finite because there is a one-one and onto correspondence between* s *and the set* $N_4 = \{1, 2, 3, 4\}$ *which is* $(1, 1)\,(\frac{1}{2}, 2)\,(c, 3)\,(99, 4)$.

Remark 5.
- *Every subset of a finite set is a finite.*
- *Cardinal of a set s is the number of elements in the set denoted by $\overline{\overline{s}}$.*
- *If A and B are finite disjoint sets, then $A \cup B$ is finite and $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}}$.*
- *If A and B are finite set, then $A \cup B$ is finite and $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}$.*
- *If $A_1, A_2, \dots, A_n$ are finite sets, then $\bigcup\limits_{i=1}^{n} A_i$ is finite.*

Definition 9. *A set X is called denumerable with cardinality $\varkappa_0$ if and only if if it is equivalent to $\mathbb{N}$, e.g., $\mathbb{Z}, \mathbb{N}$, $\mathbb{Q}$, the even integers.*

Example 9. *The sets $\mathbb{N} = \{1, 2, \dots\}$ and $E = \{2, 4, \dots\}$ are equivalent since the function*

$$f : \mathbb{N} \to E,$$

defined by $f(x) = 2x$ is one-one and onto, so the set E is denumerable.

Remark 6.
- *A set is countable if it is denumerable or finite.*
- *Every subset of a countable set is countable.*
- *The union of countable family of countable sets is countable.*
- *The union of denumerable family of countable sets is countable.*
- *If a set is not countable, then we say it is uncountable, e.g., $\mathbb{R}$, (a, b).*

Lecture Three

2 Topological Space

Definition 10. *Let X be a nonempty set. Then $\tau \subseteq \mathcal{P}(X)$ is called a topology on X if the following conditions hold:*

1. ϕ, $X \in \tau$. .
2. $G, H \in \tau \Rightarrow G \cap H \in \tau$ *,i.e., closed under finite intersection..*
3. *If $\{G_\alpha : \alpha \in \delta\}$ is a family of elements of τ. Then $\bigcup\limits_{\alpha \in \triangle} G_\alpha \in \tau$,i.e., any arbitrary union of the members of τ is an element of τ..*

The pair (X, τ) is called a topological space and the elements of τ are called open sets.

Example 10. *Let $X = \{1, 2, 3\}$, then*

$$\mathcal{P}(X) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\{1, 2, 3\}, \phi\}.$$

- $\tau_1 = \{\phi, X\}$ *is a topology on X since the conditions 1, 2, and 3 of Definition 10 hold.*
- $\tau_2 = \mathcal{P}(X)$ *is a topology on X.*
- $\tau_3 = \{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$ *is a topology on X.*
- $\tau_4 = \{\phi, X, \{2\}, \{1, 2\}, \{3\}\}$ *is not a topology on X since $\{2\}, \{3\} \in \tau_4$ but $\{2, 3\} \notin \tau_4$, i.e., Condition 3 of Definition 10 doesn't hold.*

Example 11. *Let $X = \{a, b, c\}$. Then*

- $\tau_1 = \{X, \phi, \{a\}\}$ *is a topology on X.*
- $\tau_2 = \{X, \phi, \{b\}\}$ *is a topology on X.*
- $\tau_1 \cup \tau_2$ *is not a topology on X, since $\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 \cup \tau_2$.*

Remark 7. *The union of topologies need not be a topology.*

Example 12. *Let $\mathbb{N} = \{1, 2, \dots\}$ and we define*

$$\tau = \{\{1, 2, \dots, n\} : n \in \mathbb{N}\} \cup \{\phi, \mathbb{N}\}.$$

Show that τ is a topology on $\mathbb{N}$.
To show that τ is a topology on $\mathbb{N}$, We need to verify conditions 1,2, and 3 of Definition 10.

- *Clearly $\phi, \mathbb{N} \in \tau$.*

- *Let* $G, H \in \tau$. *Then* $G = \phi\, and\, H = \phi$ *or* $G = \phi\, and\, H = \mathbb{N}$ *or* $G = \mathbb{N}\, and\, H = \mathbb{N}$ *in all these cases* $G \cap H \in \tau$. *Now, Let*

$$G = \{\{1, 2, \ldots, n\} : n \in \mathbb{N}\},$$

and

$$H = \{\{1, 2, \ldots, m\} : m \in \mathbb{N}\},$$

$$G \cap H = \{\{1, 2, \ldots, k\} : k \in \mathbb{N}\} \in \tau\, where\, k = min(n, m).$$

- *Let* $\{G_\alpha : \alpha \in \triangle\}$. *Then*

$$G_\alpha = \{1, 2, \ldots, n_\alpha\}, \alpha \in \triangle,$$

where

$$\bigcup_{\alpha \in \triangle} G_\alpha = \{1, 2, \ldots, M\},$$

and $M = \sup\{n_\alpha : \alpha \in \triangle\}$, *i.e.,*

$$\bigcup_{\alpha \in \triangle} G_\alpha \in \tau.$$

Thus $(\mathbb{N}, \tau)$ *is a topological space.*

Definition 11. *Let* (X, τ) *be a topological space and let* $F \subseteq X$. *Then* F *is a closed set if and only if* $F^c = X \setminus F$ *is open, i.e.,* $F^c \in \tau$.

Example 13. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\{b, c, d\}\}$ *be a topology on* X. *The open sets are* $X, \phi, \{a\}, \{c, d\}, \{a, c, d\}$, *and*$\{b, c, d\}$, *where the closed sets are* $\{\phi, X, \{b, c, d\}, \{a, b\}, \{b\}\}$.

Note that there are subsets of X which are both open and closed. Also, there are sets which are neither open nor closed.

Corollary 1. *Let* (X, τ) *be a topological space, then*

1. ϕ *and* X *are closed sets.*

2. *The intersection of any number of closed sets is closed.*

   Proof. Let $\{F_i : i \in I\}$ be a family of closed sets. $(\bigcap_{i \in I} F_i)^c = \bigcup_{i \in I} F_i^c$ is open since the union of an arbitrary number of open sets is open. □

3. *The finite union of the closed sets is closed.*

   Proof. Let $F_1, \ldots, F_n$ be closed sets. $(\bigcup_{i=1}^{n} F_i)^c = \bigcap_{i=1}^{n} F_i^c$ is open since the intersection of a finite number of open sets is open. □

Theorem 2. *If $\{\tau_\alpha : \alpha \in \triangle\}$ is a family of topologies on a set X, then*

$$\bigcap_{\alpha\in\triangle} \tau_\alpha \, is\, a\, topology\, on\, X.$$

Proof. Since τ_α is a topology,

$$\phi, X \in \tau_\alpha,\ \forall\alpha \in \triangle,$$

i.e., $\phi, X \in \bigcap_{\alpha\in\triangle} \tau_\alpha$*.
Let

$$G, H \in \bigcap_{\alpha\in\triangle} \tau_\alpha$$

$$G, H \in \tau_\alpha,\ \forall\alpha \in \triangle,$$

again since τ_α is a topology $G \cap H \in \tau_\alpha,\ \forall\alpha \in \triangle$ i.e., $G \cap H \in \bigcap_{\alpha\in\triangle} \tau_\alpha **$.
Let $\{G_i : i \in I\} \subseteq \bigcap_{\alpha\in\triangle} \tau_\alpha$, then $G_i \in \bigcap_{\alpha\in\triangle} \tau_\alpha$, i.e., $G_i \in \tau_\alpha \forall\alpha \in \triangle, \forall i \in I$, again since τ_α is a topology we have that $\bigcup_{i\in I} G_i \in \tau_\alpha, \forall\alpha \in \triangle$, i.e., $\bigcup_{i\in I} G_i \in \bigcap_{\alpha\in\triangle} \tau_\alpha$***
From *, **, and *** we conclude that $\bigcap_{\alpha\in\triangle} \tau_\alpha$ is a topology on X which completes the proof.

□

Remark 8. *If X is a finite set and τ is a topology on X, then we call (X, τ) a finite topological space.*

2.1 Exercises

1. Let $X = \{a, b, c\}$ and let $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on X. Find the open and the closed sets of (X, τ).

2. State if the following statement is true or false "Let τ and τ' be two topologies on the same set X. Then $\tau \cup \tau'$ is a topology on X", explain your answer.

3. Let
$$\tau = \{n, n+1, n+2, \dots\}, n \in \mathbb{N} \cup \{\phi, \mathbb{N}\}.$$
Prove that τ is topology on $\mathbb{N}$.

Lecture Four

3 Some Important Topological Spaces

3.1 The Indiscrete Topology

Let X be a nonempty set, then $\tau_{ind} = \{X, \phi\}$ is a topology on X called the indiscrete topology. The open and closed sets in τ_{ind} are ϕ, and X.

3.2 The Discrete Topology

Let X be a nonempty set, then $\tau_{dis} = \mathcal{P}(x)$ is a topology on X called the discrete topology. **Note that** all subsets of X are both open and closed.

Remark 9. *If X is a singleton set, then $\tau_{disc} = \tau_{ind}$ and these are the only topologies on X.*

3.3 The Cofinite Topology

Let X be a non empty set, we define the cofinite topology τ_{cof} to be the collection of subsets $G \subseteq X$ whose complement is finite together with the empty set ϕ ,i.e.,

$$\tau_{cof} = \{G \subseteq X : G^c = X \setminus G \, \text{is finite}\} \cup \phi.$$

To show τ_{cof} is a topology on X, we need to verify the following:

- $\phi \in \tau_{cof}$ and $X^c = X \setminus X = \phi$ which is finite.
- Let $G, H \in \tau_{cof}$, i.e., $G \subseteq X : G^c = X - G$ is finite and $H \subseteq X : H^c = X - G$ is finite . Now $G \cap H \subseteq X : (G \cap H)^c = G^c \cup H^c$ which is finite since the union of two finite sets is finite. Thus $G \cap H \in \tau_{cof}$.
- Let $\{G_i : i \in I\} \subseteq T_{cof}$, $(\bigcup\limits_{i \in I} G_i)^c = \bigcap\limits_{i \in I} G_i^c$ is finite since the intersection of finite sets is finite. Thus $\bigcup\limits_{i \in I} G_i \in \tau_{cof}$. So, (X, τ_{cof}) is a topological space.

Example 14. *Let $X = \mathbb{R}$ and*

$$\tau_{cof} = \{G \subseteq \mathbb{R} : G^c = X \setminus G \, is \, finite\}.$$

Decide if the following sets are open, close, or neither.

- $\mathbb{R} \setminus \{1\}$ *is open since its complement is finite.*
- $\mathbb{R} \setminus (0, 1)$ *is not open since its complement* $[0, 1]$ *is not finite and not closed since its complement is not open.*
- $\{0, -1, 5, 15\}$ *is closed set by Remark 11.*

- $\mathbb{N}$ *is not open since its complement is not finite and not closed since its complement is not open.*

- $\mathbb{R} \setminus \{1, 2, \ldots, 100\}$ *is open since its complement is finite.*

Remark 10. *Let X be a set and τ_{cof} is a topology on X. τ_{cof} is the discrete topology if and only if X is a finite set.*
For the topological space $(X, \tau_{cof})$and X is infinite, the open sets are ϕ, X, $X \setminus \{x_1, \ldots, x_n\}$ and the closed sets are $X, \phi, \{x_1, x_2, \ldots, x_n\}$.

Remark 11. *Every finite subset of X is closed in (X, τ_{cof}) where X is an infinite set. The set $F = \{x_1, x_2, \ldots, x_n\}$ is closed in τ_{cof}, since $F^c = X \setminus \{x_1, x_2, \ldots, x_n\}$ is open since $(F^c)^c = \{x_1, x_2, \ldots, x_n\}$ is finite.*

3.4 Cocountable Topology

Let X be a set, we define the cocountable topology τ_{coc} to be be the collection of subsets $G \subseteq X$ whose complement is countable together with the empty set ϕ, i.e.,

$$\tau_{coc} = \{G \subseteq X : G^c = X \setminus G \ \ is\, countable\} \cup \phi.$$

To show τ_{cof} is topology we need to verify the following:

- $\phi \in \tau_{coc}$ and $X^c = X \setminus X = \phi$ which is countable.

- Let $G, H \in \tau_{coc}$, i.e., $G \subseteq X : G^c = X - G$ is countable and $H \subseteq X : H^c = X - G$ is countable . Now $G \cap H \subseteq X : (G \cap H)^c = G^c \cup H^c$ which is countable since the union of two countable sets is countable. Thus $G \cap H \in \tau_{coc}$.

- Let $\{G_i : i \in I\} \subseteq T_{coc}$, $(\bigcup\limits_{i \in I} G_i)^c = \bigcap\limits_{i \in I} G_i^c$ is countable since the intersection of countable set is countable. Thus $\bigcup\limits_{i \in I} G_i \in \tau_{cof}$. So, (X, τ_{coc}) is a topological space.

Remark 12. *Every countable subset of X is closed in (X, τ_{coc}), where X is infinite.*

Example 15. *Let $X = \mathbb{R}$ and*

$$\tau_{coc} = \{G \subseteq \mathbb{R} : G^c = X \setminus G\, is\, countable\}.$$

Decide if the following sets are open, closed, or neither.

- $\{1, 2, 3\}$ *is closed since its complement is open.*

- $\mathbb{N}$ *is closed since its complement is open.*

- $\mathbb{R} \setminus \{0, -1, 5, 15\}\}$ *is open since its complement is countable.*

- $\mathbb{Q}$ *is closed since it's a countable set.*

- $[0, 7]$ *is neither open nor closed.*

3.5 Exercises

1. Let $\mathbb{R}$ be the set of real numbers and let

$$\tau_{coc} = \{G \subseteq \mathbb{R} : G^c = \mathbb{R} \setminus G \, \text{is countable}\} \cup \phi.$$

   Show that τ_{coc} is a topology on $\mathbb{R}$.

2. Is there a set upon which the discrete and indiscrete topologies are equal.

3. State if the following statement is **True** or **False** and justify your answer
   Let $(\mathbb{R}, \tau_{cof})$ be a topological space. Any finite set is open in τ_{cof}.

Lecture Four

3.6 The Left Ray Topology

Let $X = \mathbb{R}$. We define

$$\tau_\ell = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}.$$

Now τ_ℓ is topology on $\mathbb{R}$. To verify this we will check Conditions 1, 2, and 3 of Definition 10.

- It is clear that $\phi, \mathbb{R} \in \tau_\ell$.
- Let $G, H \in \tau_\ell$, then $G = (-\infty, r) : r \in \mathbb{R}$ and $H = (-\infty, m) : m \in \mathbb{R}$, $G \cap H = (-\infty, k)$, *where* $k = \min\{r, s\}$.
- Let $\{G_\alpha : \alpha \in \triangle\} \subseteq \tau_\ell$, then $G_\alpha = (-\infty, r_\alpha), \forall \alpha \in \triangle$, $\bigcup\limits_{\alpha \in \triangle} G_\alpha = \bigcup\limits_{\alpha \in \triangle} (-\infty, r_\alpha) = (-\infty, M) \in \tau_\ell$, where $M = \sup\{r_\alpha : \alpha \in \triangle\}$. So, τ_ℓ is a topology on $\mathbb{R}$.

Example 16. *In (X, τ_ℓ), decide if the following sets are open, closed or neither*

- $(-\infty, 7)$ *is an open set.*
- $[7, \infty)$ *is a closed set. since* $[7, \infty)^c = (-\infty, 7)$ *is open set.*
- $(0, 5)$ *neither.*
- $(2, \infty)$ *not closed, not open.*
- $\{1, 2, 3\}$ *neither open nor closed.*

3.7 The Right Ray Topology

Let $X = \mathbb{R}$. We define

$$\tau_r = \{(s, \infty) : s \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}.$$

Now τ_r is topology on $\mathbb{R}$. To verify this we will check Conditions 1, 2, and 3 of Definition 10.

- It is clear that $\phi, \mathbb{R} \in \tau_r$.
- Let $G, H \in \tau_r$, then $G = (s, \infty) : s \in \mathbb{R}$ and $H = (k, \infty) : k \in \mathbb{R}$, $G \cap H = (t, \infty) \in \tau_r$, *where* $t = \max\{s, k\}$.
- Let $\{G_\alpha : \alpha \in \triangle\} \subseteq \tau_r$, then $G_\alpha = (r_\alpha, \infty), \forall \alpha \in \triangle$, $\bigcup\limits_{\alpha \in \triangle} G_\alpha = \bigcup\limits_{\alpha \in \triangle} (s_\alpha, \infty) = (M, \infty) \in \tau_r$, where $M = \inf\{s_\alpha : \alpha \in \triangle\}$. So, τ_r is topology on $\mathbb{R}$.

3.8 The Usual Topology

Open sets in $\mathbb{R}$

Definition 12. *Let A be a set of real numbers. A point $x \in A$ is an interior point of A if and only if x belongs to some open interval S_x which is contained in A, i.e., $\forall x \in A; \exists S_x = (a, b) : x \in (a, b) \subseteq A$.*

Definition 13. *The set A is open if and only if each of its points is an interior point.*

Example 17. *An open interval (a,b) is an open set in $\mathbb{R}$, since $\forall x \in (a, b)$, $\exists S_x : x \in S_x \subseteq (a, b)$.*

Example 18. *The closed interval $B = [a, b]$ is not open set in $\mathbb{R}$, since any open interval containing a or b must contain points outside of B. Hence the end points a and b are not interior points of B. Also, any singlton $\{x\}$ is not an open set in $\mathbb{R}$.*

Theorem 3.

1. *The union of any number of open sets in $\mathbb{R}$ is open.*
2. *The intersection of any finite number of open sets in $\mathbb{R}$ is open.*

Remark 13. *The finiteness condition in 2 of Theorem 3 can not be removed, if we consider $A_n = \{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$, i.e., $\bigcap_{n=1}^{\infty} (-1, 1), (\frac{-1}{2}, \frac{1}{2}), \cdots = \{0\}$ which is not open in $\mathbb{R}$.*

The usual topology

Let $X = \mathbb{R}$ and we define

$$\tau_u = \{G \subseteq \mathbb{R} : \forall x \in G, \exists\, an\, open\, interval\, (a, b) : x \in (a, b) \subseteq G\} \cup \phi.$$

τ_u is a topology on $\mathbb{R}$.

Open and closed sets in τ_u.

- Every open interval is an open set.
- Every closed interval is a closed set since its complement is open.
- Every singleton set is closed since its complement is a union of open sets.
- Every finite set is closed since its complement is a union of open sets.
- The interval $[a, b)$ is not open and not closed.
- The interval $[a, \infty)$ not open but closed.

4 Comparison of Topologies

Definition 14. *let τ_1, τ_2 be topologies on the non empty set X. Suppose that each open set in τ_1 is open in τ_2 , that is τ_1 is a subclass of τ_2, i.e., $\tau_1 \subseteq \tau_2$. In this case we say that τ_1 is coarser or smaller (weaker) than τ_2 or τ_2 is finer or larger than τ_1.*

Remark 14. *We say that two topologies are not comparable if neither of them is coarser than the other.*

Example 19. *Consider τ_{dis}, τ_{ind} and any other topology τ on $\mathbb{R}$. τ_{dis} is finer than τ and τ is weaker than τ_{dis}. Also, τ_{ind} is weaker than τ and τ is finer than τ_{ind}*

Example 20. *Consider the topological spaces $(\mathbb{R}, \tau_u)$ and $(\mathbb{R}, \tau_{cof})$. Compare τ_u with τ_{cof}.*
Every open set in τ_{cof} has the form ϕ, $\mathbb{R}, \mathbb{R} \setminus \{a_1, \ldots, a_n\}$ which are elements in τ_u, i.e., τ_{cof} is weaker than τ_u. Note that $(0, 1)$ is open in τ_u but not open in τ_{cof} since its complement is not finite. So, τ_u is finer than τ_{cof}.

Example 21. *Let $X = \{a, b\}$ be a 2-element set and $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$ are two topologies on X that are not comparable.*

4.1 Exercises

1. Compare the cofinite topology with
   - cocountable topology.
   - left ray topology.
2. Give an example of a collection of open sets in the space $(\mathbb{R}, \tau_u)$ whose intersection is not open.
3. State if the following statement is **True** or **False** and justify your answer. The interval $(-\infty, 3]$ is open in τ_u.
4. Define the following topological space and describe the closed and open sets in each space where X is a nonempty set.
   - (X, τ_{dis}).
   - (X, τ_{ind}).
   - $(\mathbb{R}, \tau_\ell)$.
   - $(\mathbb{R}, \tau_r)$.
   - $(\mathbb{R}, \tau_{cof})$.
   - $(\mathbb{R}, \tau_{coc})$.
   - $(\mathbb{R}, \tau_u)$.

Lecture Six

5 Topology Induced by Function

Recall Let $f : X \to Y$ be a function. Let $A \subseteq X$ and $C \subseteq Y$. Then

- The image (or range) of A denoted by $f(A)$ is defined as
$$f(A) = \{f(x) : x \in A\}.$$
- The inverse image (or the preimage) of C denoted by $f^{-1}(C)$ is defined as
$$f^{-1}(C) = \{x \in X : f(x) \in C\}.$$

Remark 15.
$f(A) \subseteq Y$.
$f^{-1}(C) \subseteq X$.
$f^{-1}(\phi) = \phi$.
$f^{-1}(Y) = X$.

Theorem 4. *Let I be an arbitrary set. Let $f : X \to Y$ be a function. Assume $A_i \subseteq X$ and $B_i \subseteq Y$ for all $i \in I$. Then*

- $f(\bigcup\limits_{i\in I} A_i) = \bigcup\limits_{i\in I}(f(A_i))$.
- $f(\bigcap\limits_{i\in I} A_i) \subseteq \bigcap\limits_{i\in I}(f(A_i))$.
- $f^{-1}(\bigcup\limits_{i\in I} B_i) = \bigcup\limits_{i\in I}(f^{-1}(B_i))$.
- $f^{-1}(\bigcap\limits_{i\in I} B_i) = \bigcap\limits_{i\in I}(f^{-1}(B_i))$.
- $f^{-1}(B^c) = [f^{-1}(B)]^c$.
- $f(f^{-1}(B)) \subseteq B$.
- $A \subseteq f^{-1}(f(A))$.

Theorem 5. *Let $f : X \to Y$ be a function and suppose that X has a topology τ_x. Then the collection*
$$\tau_y = \{V \subseteq Y : f^{-1}(V) \in \tau_x\} \subseteq \mathcal{P}(Y),$$
is atopology on Y called the induced topology by the function f and the topological space (X, τ_x).

Proof. We need to verify conditions 1, 2, and 3 of Definition 10.

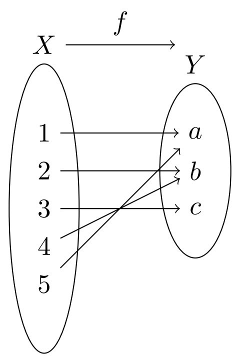
- $\phi \subseteq Y : f^{-1}(\phi) = \phi \in \tau_x$ so $\phi \in \tau_y$. Also, $Y \subseteq Y : f^{-1}(Y) = X \in \tau_x$. Thus condition 1 of Definition 10 holds.
- Let $U, V \in \tau_y$, i.e., $U \subseteq Y : f^{-1}(U) \in \tau_x$ and $V \subseteq Y : f^{-1}(V) \in \tau_x$. Now $U \cap V \subseteq Y : f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \in \tau_x$. Thus, $U \cap V \in \tau_y$.
- Let $\{G_i : i \in I\} \subseteq \tau_y$, this means $G_i \subseteq Y : f^{-1}(G_i) \in \tau_x$, $i \in I$. Now $\bigcup\limits_{i\in I} G_i \subseteq Y : f^{-1}(\bigcup\limits_{i\in I} G_i) = \bigcup\limits_{i\in I} f^{-1}(G_i) \in \tau_x$ since it is an arbitrary union of open sets in τ_x. Thus, τ_y is a topology on Y which completes the proof.

$\square$

Example 22. *Let $X = \{1, 2, 3, 4, 5\}$ and let*

$$\tau_x = \{\phi, X, \{1\}, \{2, 4\}, \{3, 5\}, \{1, 2, 4\}, \{1, 3, 5\}, \{2, 3, 4, 5\}\}$$

be a topology on X. If $Y = \{a, b, c\}$ and



Find τ_y induced by f and the topological space (X, τ).
From the definition of $\tau_y = \{V \subseteq Y : f^{-1}(V) \in \tau_x\}$, we should find the power set of Y and test if the inverse image of each element in $\mathcal{P}(Y)$ is open in τ_x.

$\mathcal{P}(Y)$	$f^{-1}(U) : U \in \mathcal{P}(Y)$	*Elements of τ_y*
Y	$f^{-1}(Y) = X \in \tau_x$	$Y \in \tau_y$
ϕ	$f^{-1}(\phi) = \phi \in \tau_x$	$\phi \in \tau_y$
$\{a\}$	$f^{-1}\{a\} = \{1, 5\} \notin \tau_x$	$\{a\} \notin \tau_y$
$\{b\}$	$f^{-1}\{b\} = \{2, 4\} \in \tau_x$	$\{b\} \in \tau_y$
$\{c\}$	$f^{-1}\{c\} = \{3\} \notin \tau_x$	$\{c\} \notin \tau_y$
$\{a, b\}$	$f^{-1}\{a, b\} = \{4, 2, 1, 5\} \notin \tau_x$	$\{a, b\} \notin \tau_y$
$\{a, c\}$	$f^{-1}\{a, c\} = \{1, 5, 3\} \in \tau_x$	$\{a, c\} \in \tau_y$
$\{b, c\}$	$f^{-1}\{b, c\} = \{2, 3, 4\} \notin \tau_x$	$\{b, c\} \notin \tau_y$

Now, $\tau_y = \{Y, \phi, \{b\}, \{a, c\}\}$.

Theorem 6. *Let* $f : X \to Y$ *and let* Y *have a topology* τ_y. *Then the collection*

$$\tau_x = \{f^{-1}(U) : U \in \tau_y\} \subseteq \mathcal{P}(x)$$

is a topology on X *called the induced topology by the function* f *and the topological space* (Y, τ_y).

Proof. We need to verify conditions 1, 2, and 3 of Definition 10.

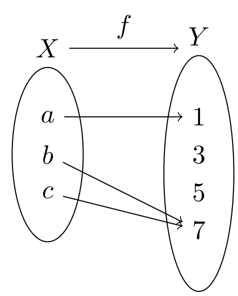
- $f^{-1}(\phi) = \phi$, $\phi \in \tau_y$ so, $\phi \in \tau_x$. Also, $f^{-1}(Y) = X, Y \in \tau_y$ so, $X \in \tau_x$.
- Let $A_1, A_2 \in \tau_x$, i.e., $\exists U_1 \in \tau_y : A_1 = f^{-1}(U_1)$ and $\exists U_2 \in \tau_y : A_2 = f^{-1}(U_2)$. Now, $A_1 \cap A_2 = f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2)$ but $U_1 \cap U_2 \in \tau_y$. Thus $A_1 \cap A_2 \in \tau_x$.
- Let $\{G_i : i \in I\} \subseteq \tau_x$, i.e., $\exists U_i \in \tau_y : f^{-1}(U_i) = G_i, \forall i \in I$. Now, $\bigcup\limits_{i \in I} G_i = \bigcup\limits_{i \in I} f^{-1}(U_i) = f^{-1}(\bigcup\limits_{i \in I} U_i)$ but $\bigcup\limits_{i \in I} U_i \in \tau_y$. Thus $\bigcup\limits_{i \in I} G_i \in \tau_x$. Since conditions 1, 2, and 3 of Definition 10 hold, τ_x is a topology on X and this completes the proof.

□

Example 23. *Let* $X = \{a, b.c\}$ *and* $Y = \{1, 3, 5, 7\}$. *We define the topology* τ_y *on* Y *as*

$$\tau_y = \{\phi, Y, \{1, 5\}, \{5, 7\}, \{5\}, \{1, 5, 7\}\}$$

Find τ_x *which induced by the topological space* (X, τ) *and the function* $f : X \to Y$



As we know $\tau_x = \{f^{-1}(U) : U \in \tau_y\}$, *so we need to find the inverse image of open sets in* τ_u.

$U \in \tau_y$	$f^{-1}(U) \in \tau_x$
$\phi \in \tau_y$	$f^{-1}(\phi) = \phi \in \tau_x$
$Y \in \tau_y$	$f^{-1}(Y) = X \in \tau_x$
$\{5, 7\} \in \tau_y$	$f^{-1}(\{5, 7\}) = \{b, c\} \in \tau_x$
$\{5\} \in \tau_y$	$f^{-1}(\{5\}) = \phi \in \tau_x$
$\{1, 5, 7\} \in \tau_y$	$f^{-1}(\{5, 7, 1\}) = \{a, b, c\} = X \in \tau_x$
$\{1, 5\} \in \tau_y$	$f^{-1}(\{1, 5\}) = \{a\} \in \tau_x$

Thus $\tau_x = \{\phi, X, \{b, c\}, \{a\}\}$.

6 Subspace

Remark 16. *Let $(A_i)_{i\in I}$ be a family of subsets of X, where $I$is an arbitrary set. Let $B \subseteq X$. Then*

- $B \cap (\bigcup\limits_{i\in I} A_i) = \bigcup\limits_{i\in I}(B \cap A_i)$.
- $B \cup (\bigcap\limits_{i\in I} A_i) = \bigcap\limits_{i\in I}(B \cup A_i)$.

Definition 15. *Let (X, τ) be a topological space $A \subseteq X$. Then*

$$\tau_\alpha = \{G \cap A : G \in \tau\}$$

is called the relative (induced) topology on A and (A, τ_α) is called the subspace of (X, τ).

Theorem 7. *τ_α is a topology on A.*

Proof.

- $\phi \in \tau_\alpha$ since $\phi \cap A = \phi : \phi \in \tau$. Also, $A \in \tau_\alpha$ since $X \cap A = A : X \in \tau$, note that $A \subseteq X$, so $A \cap X = A$.
- Let $H_1, H_2 \in \tau_\alpha$, i.e., $\exists G_1 \in \tau : G_1 \cap A = H_1$, $\exists G_2 \in \tau : G_2 \cap A = H_2$. Now $H_1 \cap H_2 = G_1 \cap A \cap G_2 \cap A = (G_1 \cap G_2) \cap A \, and \, (G_1 \cap G_2) \in \tau \, as \, \tau$ is a topology which is closed under finite intersection. Thus $H_1 \cap H_2 \in \tau_\alpha$.
- Let $\{H_i : i \in I\} \subseteq \tau_\alpha$, i.e., $\exists G_i \in \tau : G_i \cap A = H_i, \forall i \in I$. Now, $\bigcup\limits_{i\in I} H_i = \bigcup\limits_{i\in I}(G_i \cap A) = A \cap (\bigcup\limits_{i\in I} G_i)$ but $\bigcup\limits_{i\in I} G_i \in \tau$ as τ is a topology which closed under arbitrary union, so $\bigcup\limits_{i\in I} H_i \in \tau_\alpha$. Thus τ_α is a topology on A and this completes the proof.

$\square$

6.1 Exercises

1. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$. We define

   $$\tau_x = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\},$$

   to be a topology on X. Find the topology on Y induced by the topological space (X, τ_x) and the following function

   $$f = \{(1, a), (2, a), (3, d), (4, b)\}$$

2. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$. We define

   $$\tau_y = \{Y, \phi, \{c\}, \{a, b, c\}, \{c, d\}\},$$

   to be a topology on Y. Find the topology on X induced by the topological space (Y, τ_y) and the following function

   $$f = \{(1, a), (2, c), (3, c), (4, d)\}$$

3. Let $X = \{a, b, c, d, e\}$. We define

$$\tau = \{X, \phi, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\},$$

to be a topology on X. Let $A = \{a, d, e\}$ be a subset of X. Find the relative (induced) topology on A.

Lecture Seven

Example 24. *Let* $X = \{a, b, c, d, e\}$. *We define*

$$\tau = \{X, \phi, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\},$$

to be a topology on X. *Let* $A = \{a, d, e\}$ *be a subset of* X. *Find the relative (induced) topology on* A.

$G \in \tau$	$G \cap A$
$\phi \in \tau$	$\phi \cap A = \phi$
$X \in \tau$	$X \cap A = A$
$\{c, d\} \in \tau$	$\{c, d\} \cap A = \{d\}$
$\{a, c, d\} \in \tau$	$\{a, c, d\} \cap A = \{a, d\}$
$\{b, c, d, e\} \in \tau$	$\{b, c, d, e\} \cap A = \{d, e\}$

$\tau_\alpha = \{A, \phi, \{d\}, \{a, d\}, \{d, e\}\}$.

Example 25. *Consider the topological space* $(\mathbb{R}, \tau_u)$ *and the set* $A = [-1, 1]$. *State which of the following is open in* $\tau_\alpha = \{G \cap A : G \in \tau\}$.

- $(\frac{1}{2}, 1)$ *is open in* τ_α *since* $\exists(\frac{1}{2}, 1) \in \tau : (\frac{1}{2}, 1) \cap [-1, 1] = (\frac{1}{2}, 1)$.
- $(\frac{1}{2}, 1]$*is open in* τ_α *since* $\exists(\frac{1}{2}, 2) \in \tau : (\frac{1}{2}, 2) \cap [-1, 1] = (\frac{1}{2}, 1]$.
- $[\frac{1}{2}, 1)$*is not open in* τ_α *since there is no* $U \in \tau : U \cap [-1, 1] = (\frac{1}{2}, 1]$ *as the point* $\frac{1}{2}$ *must be an interior point of* U *so,* $\forall \epsilon > 0$ $U = (\frac{1}{2} - \epsilon, 1)$ *and it is not possible that* $U \cap [-1, 1] = [\frac{1}{2}, 1)$.

Example 26. *Consider the topological space* $(\mathbb{R}, \tau_\ell) = \{(-\infty, a) : a \in \mathbb{R}\}$ *and the set* $A = [0, 1]$. *Describe* $\tau_\alpha = \{G \cap A : G \in \tau\}$.

- $a < 0$, *then* $A \cap (-\infty, a) = \phi$.
- $0 < a < 1$, *then* $A \cap (-\infty, a) = [0, a)$.
- $a > 1$, *then* $A \cap (-\infty, a) = A$.

Thus $\tau_\alpha = \{\phi, A, [0, a)\}$ *where* $0 < a < 1$.

Theorem 8. *Let* A *be an open subset of the space* (X, τ). *Then* $\tau_\alpha \subseteq \tau$, *i.e., every open set in* τ_α *is open in* τ.

Proof. Let $H \in \tau_\alpha$ and $A \in \tau$, $\exists G \in \tau : A \cap G = H$ but $A \in \tau$ so, $A \cap G \in \tau$, i.e., $H \in \tau$. $\square$

7 Closure of the Set

Definition 16. *Let (A, X) be a topological space and $A \subseteq X$. Then the closure of A is the intersection of closed sets in X which contain A and is denoted by $\overline{A}$, i.e.,*

$$\overline{A} = \bigcap\{F : F \text{ is closed and } A \subseteq F\}.$$

Example 27. *Let $X = \{a, b, c, d, e\}$, we define $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ to be topology on X. It is easy to see that the closed sets are $\{X, \phi, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}\}$. We can find*

- $\overline{\{b\}} = \bigcap\{F : F \text{ is closed and } A \subseteq F\} = X \cap \{b, c, d, e\} \cap \{b, e\} = \{b, e\}$.
- $\overline{\{a, c\}} = X$.
- $\overline{\{b, d\}} = \{b, c, d, e\}$.

Remark 17.

1. *$\overline{A}$ is closed since it is an arbitrary intersection of closed sets.*
2. *$A \subseteq \overline{A}$.*
3. *If $A \subseteq F$, F is closed, then $\overline{A} \subseteq F$.*
4. *$\overline{A}$ is the smallest closed set containing A.*
5. *A is closed iff $\overline{A} = A$.*
6. *$\overline{\overline{A}} = \overline{A}$. Since $\overline{A}$ is closed by 1 and by 5 we have $\overline{\overline{A}} = \overline{A}$.*
7. *If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

   Proof. By 2, we have $A \subseteq B \subseteq \overline{B}$, but $\overline{B}$ is closed by 4. Now use 3, we get $\overline{A} \subseteq \overline{B}$. □

8. *$\overline{\phi} = \phi$, $\overline{X} = X$. Since ϕ and X are closed sets, by 5 we get these results.*
9. *$\overline{A \cup B} = \overline{A} \cup \overline{B}$*

   Proof. We have, $A \subseteq A \cup B$ and by 7 $\overline{A} \subseteq \overline{A \cup B}*$. The same holds for B, i.e., $B \subseteq A \cup B$ and by 7 $\overline{B} \subseteq \overline{A \cup B}*$. From * we have $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} **$
   By 2, $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, $A \cup B \subseteq \overline{A} \cup \overline{B}$. By 7, $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$ but $\overline{A} \cup \overline{B}$ is a closed set since it is a union of two closed sets and by 5, $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$.
   Thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B} **$
   From ** we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$ which completes the proof. □

7.1 Exercises

1. Consider the topological space $(\mathbb{R}, \tau_u)$ and the set $A = [3, 8] \subseteq \mathbb{R}$. Decide if $[3, 5)$ is an open set in the relative topology on A.

2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ be a topology on X. Find

   - $\overline{\{b\}}$.
   - $\overline{\{c\}}$.
   - $\overline{\{a\}}$
   - $\overline{\{a, b\}}$

Lecture Eight

Example 28. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ *be a topology on* X*. Find*

- $\overline{\{b\}} = \{b, c\}$.
- $\overline{\{}c\} = \{b, c\}$.
- $\overline{\{a\}} = \{a\}$
- $\overline{\{a, b\}} = X$

Remark 18. *A subset of a topological space* X *is said to be dense in* X *iff* $\overline{A} = X$*. In Example 28. the subset* $\{a, b\}$ *is dense in* X*.*

Example 29. *In the topological space* (X, τ_{ind}) *describe the closure of any set* $A \subseteq X$.

$$\overline{A} = \begin{cases} A & if A = \phi \\ X & if A \neq \phi \end{cases}$$

Example 30. *In the topological space* (X, τ_{dis}) *describe the closure of any set* $A \subseteq X$.

$$\overline{A} = A,$$

since any set $A \subseteq X$ *is closed.*

Example 31. *In the topological space* $(\mathbb{R}, \tau_{cof})$*, the open sets are* $\mathbb{R}, \phi, \mathbb{R} \setminus \{x_1, \ldots, x_n\}$ *and the closed sets are* $\mathbb{R}, \phi, \{x_1, \ldots, x_n\}$*. Find*

- $\overline{(0, 5)} = \mathbb{R}$.
- $\overline{\{3\}} = \{3\}$.
- $\overline{(-\infty, 0)} = \mathbb{R}$.
- $\overline{\{1, 2, 4\}} = \{1, 2, 4\}$.
- $\overline{\mathbb{N}} = \mathbb{R}$.

Example 32. *In the topological space* (X, τ_u)*, the open sets are* $\mathbb{R}, \phi, (a, b)$ *and the closed sets are* $\mathbb{R}, \phi, [a, b], (-\infty, 0], \{x_1, \ldots, x_n\}$*. Find*

- $\overline{[1, 3)} = [1, 3]$.
- $\overline{(-\infty, 0)} = (-\infty, 0]$.
- $\overline{\{1, 2, 4\})} = \{1, 2, 4\}$.
- $\overline{(1, 2) \cup \{3\}} = [1, 2] \cup \{3\}$.
- $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 33. *In the topological space $(\mathbb{R}, \tau_\ell)$, the open sets are $\mathbb{R}, \phi, (-\infty, a)$ and the closed sets are $\mathbb{R}, \phi, [a, \infty)$. Find*

- $\overline{(1,2)} = [1, \infty)$.
- $\overline{(-\infty, 3)} = \mathbb{R}$.
- $\overline{\mathbb{R} \setminus \{1\}} = \overline{(-\infty, 1) \cup (1, \infty)} = \mathbb{R} \cup [1, \infty) = \mathbb{R}$.
- $\overline{(0, \infty)} = [0, \infty)$.
- $\overline{[1, \infty)} = [1, \infty)$.
- $\overline{(0, 2) \cup \{3\}} = [0, \infty) \cup [3, \infty) = [0, \infty)$.

Example 34. *In the topological space $(\mathbb{R}, \tau_{coc})$ the open sets are $\mathbb{R}, \phi, \mathbb{R} \setminus \{countable\, set\}$ and the closed sets are $\mathbb{R}, \phi, \{countable\, set\}$. Find*

- $\overline{\{1, 2, 3\}} = \{1, 2, 3\}$.
- $\overline{\mathbb{N}} = \mathbb{N}$.
- $\overline{\mathbb{Q}} = \mathbb{Q}$.
- $\overline{(0, 1)} = \mathbb{R}$.
- $\overline{[1, \infty]} = \mathbb{R}$.

Example 35.

- $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

  Proof. We have $A \cap B \subseteq A$ by 7 in Remark 17, $\overline{A \cap B} \subseteq \overline{A}*$. The same is true for B, i.e., $A \cap B \subseteq B$ by 7 in Remark 17 , $\overline{A \cap B} \subseteq \overline{B}*$. From *, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ which completes the proof. □

- *Give an example to show that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.*
  Consider the topological space (X, τ_u) and the two sets $A = [0, 1), B = (1, 2]$, $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$. $\overline{A} \cap \overline{B} = \{1\}$ and $A \cap B = \phi$ where, $\overline{\phi} = \phi$. Thus $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Lecture Nine

8 Interior of a set

Definition 17. *Let (X,τ) be a topological space and $A \subseteq X$. Then*

- *A point $x \in A$ is an interior point of A if and only if $\exists$ an open set G : $x \in G \subseteq A$. The set of all interior points of A is denoted by int(A)= A^0.*

Example 36. *Consider the topology*

$$\tau = \{X, \phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$$

on $X = \{a,b,c,d,e\}$ and the subset $A = \{b,c,d\}$ of X. To find A^0 we will test if for every point in A there exists $G \in \tau$: $x \in G \subseteq A$. The point b is not an interior point of A. The points $c,d \in A^0$ so, $A^0 = \{c,d\}$.

Remark 19.

- *The set A^0 is the largest open set contained in A, i.e., $A^0 \subseteq A$.*
- *$A^0 = \bigcup\{U \in \tau : U \subseteq A\}$.*
- *Let (X,τ) be a topological space. $A \subseteq X$ is open $\iff$ $\forall x \in A, \exists G \in \tau$: $x \in G \subseteq A$.*
- *Let (X,τ) be a topological space. $A \subseteq X$ is open $\iff$ $A^0 = A$.*

Proof. We start to prove the direction $\Rightarrow$. Suppose that $A \subseteq X$ is open, we need to show $A^0 = A$. We know that $A^0 \subseteq A$ so, we will prove $A \subseteq A^0$. Let $x \in A$. Since A is open, $\exists G \in \tau : x \in G \subseteq A$, i.e., $x \in A^0$. Thus $A = A^0$. Now, We prove the direction $\Leftarrow$. Suppose that $A^0 = A$ and $x \in A = A^0$, this means $\exists G \in \tau : x \in G \subseteq A$, i.e., A is open which completes the proof. ☐

Example 37. *Consider the set $A = (0,3] \subseteq \mathbb{R}$. Complete the following table*

$A = (0,3]$		
Toplogical space	A^0	$\overline{A}$
$(\mathbb{R}, \tau_{ind})$	ϕ	$\mathbb{R}$
$(\mathbb{R}, \tau_{dis})$	$(0,3]$	$(0,3]$
$(\mathbb{R}, \tau_{cof})$	ϕ	$\mathbb{R}$
$(\mathbb{R}, \tau_{\ell})$	ϕ	$[0,\infty)$
$(\mathbb{R}, \tau_u)$	$(0,3)$	$A = [0,3]$
$(\mathbb{R}, \tau_{coc)}$	ϕ	$\mathbb{R}$

Lecture Ten

9 Interior, Boundary and Exterior of a set

Theorem 9. *Let (x, τ) be a topological space, $A, B \subseteq X$. Then*

- $\phi^0 = \phi$, $X^0 = X$ *because* ϕ, X *are open sets.*
- $A \subseteq B \Rightarrow A^0 \subseteq B^0$.

  Proof. Suppose that $A \subseteq B$, we have $A^0 \subseteq A \subseteq B$. Let $x \in A^0$, then $\exists G \in \tau : x \in G \subseteq A \subseteq B$ this means $x \in B^0$ which completes the proof.
  Another proof: we have $A^0 \subseteq A \subseteq B$ and B^0 is the largest open set contained in B and A^0 is an open set contained in B. Thus A^0 must be $\subseteq B^0$. □

- $(A \cap B)^0 = A^0 \cap B^0$.

  Proof. We need to show $(A \cap B)^0 \subseteq A^0 \cap B^0$ and $A^0 \cap B^0 \subseteq (A \cap B)^0$. We know $A^0 \subseteq A$ and $B^0 \subseteq B$, thus $A^0 \cap B^0 \subseteq A \cap B$ but A^0, B^0 are open and so their intersection, this means $(A^0 \cap B^0)^0 = A^0 \cap B^0$. Thus $A^0 \cap B^0 \subseteq (A \cap B)^0*$.
  Also, $A \cap B \subseteq A$ implies $(A \cap B)^0 \subseteq A^0$. Similarly, $A \cap B \subseteq B$ implies $(A \cap B)^0 \subseteq B^0$. Thus $(A \cap B)^0 \subseteq A^0 \cap B^0*$. From *, we have $(A \cap B)^0 = A^0 \cap B^0$ which completes the proof. □

- $A^0 \cup B^0 \subseteq (A \cup B)^0$.

  Proof. We know $A^0 \subseteq A$ and $B^0 \subseteq B$, $A^0 \cup B^0 \subseteq A \cup B$, $(A^0 \cup B^0)^0 \subseteq (A \cup B)^0$ but $(A^0 \cup B^0)$ is open. Thus $A^0 \cup B^0 \subseteq (A \cup B)^0$ which completes the proof. □

Example 38. *Give an example to show that $A^0 \cup B^0 \neq (A \cup B)^0$.*
Consider the topological space $(\mathbb{R}, \tau_u)$ and $A = [0, 3]$, $B = [3, 5)$, $A^0 = (0, 3)$,$B^0 = (3, 5)$. Now, $A \cup B = [0, 5)$, $(A \cup B)^0 = (0, 5)$, $A^0 \cup B^0 = (0, 5) \setminus \{3\}$. Thus $A^0 \cup B^0 \neq (A \cup B)^0$.

Definition 18. *Let (X, τ) be a topological space and $A \subseteq X$. Then*

- *A point $x \in X$ is an exterior point of A if and only if $\exists$ an open set G : $x \in G \subseteq A^c$. The set of all exterior points of A is denoted by Ext(A).*
- *A point $x \in X$ is a boundary point of A if and only if $\forall$ open set $G : x \in G \Rightarrow G \cap A \neq \phi$, and$G \cap A^c \neq \phi$. The set of all boundary points of A is denoted by Bd(A).*

Example 39. *Consider the topology*

$$\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

on $X = \{a, b, c, d, e\}$ *and the subset* $A = \{b, c, d\}$ *of* X*. To find* $Ext(A)$ *we will test if for every point in* X *there exists* $G \in \tau$*:* $x \in G \in A^c$*. Only the point* $a \in Ext(A)$ *so,* $Ext(A) = \{a\}$*. Now to find* $Bd(A)$ *we will test if for every point in* X *and for every open set* $G : x \in G$*,* $G \cap A \neq \phi$ *and* $G \cap A^c \neq \phi$*. The points* $b, e \in Bd(A)$ *so,* $Bd(A) = \{b, e\}$*.*

Remark 20.

- *For any subset* $A \subseteq X$*, the sets* $int(A), Ext(A)$*, and* $Bd(A)$ *are disjoint.*
- *From Definitions 17, 18 it is clear that* $A^0 \subseteq A$*,* $Ext(A) \subseteq A^c$*.*
- *From Definition 17, we get that* $x \in Ext(A)$ *if and only if* $\exists G \in \tau : x \in G \subseteq A^c$ *if and only if* $x \in int(A^c)$*. That is* $Ext(A) = int(A^c)$*.*
- *If* $A \subseteq X$*,* $x \in X$*, then* x *belongs to one and only one of the sets* $A^0, Ext(A), Bd(A)$*.*
- $A^0 \cup Ext(A) \cup Bd(A) = X$*.*

Example 40. *Consider the set* $A = (0, 3] \subseteq \mathbb{R}$*. Complete the following table*

$A = (0,3]$				
Toplogical space	A^0	$Ext(A)$	$bd(A)$	$\overline{A}$
$(\mathbb{R}, \tau_{ind})$	ϕ	ϕ	$\mathbb{R}$	$\mathbb{R}$
$(\mathbb{R}, \tau_{dis})$	$(0,3]$	$\mathbb{R} \setminus (0,3]$	ϕ	$(0,3]$
$(\mathbb{R}, \tau_{cof})$	ϕ	ϕ	$\mathbb{R}$	$\mathbb{R}$
$(\mathbb{R}, \tau_{\ell})$	ϕ	$(-\infty, 0)$	$[0, \infty)$	$[0, \infty)$
$(\mathbb{R}, \tau_u)$	$(0,3)$	$(-\infty, 0) \cup (3, \infty)$	$\{0, 3\}$	$[0, 3]$
$(\mathbb{R}, \tau_{coc)}$	ϕ	ϕ	$\mathbb{R}$	$\mathbb{R}$

HW *try to complete the table with* $A = [0, 1]$*.*

9.1 Exercises

1. Let $X = \{a, b, c\}$ and let $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on X. Find (**Show the details of your work**)
   - The closed subsets of (X, τ)
   - $\overline{\{a\}}$
   - $\{b, c\}^0$
   - $Bd(\{b, c\})$

- $Ext(\{b, c\})$

2. Let
$$\tau = \{\{n, n+1, n+2, \dots\} : n \in \mathbb{N}\} \cup \phi.$$
be a topology on $\mathbb{N}$.

   - Find the interior points of the set $A = \{4, 13, 28, 37\}$.
   - Find the closure of the set $A = \{7, 24, 47, 85\}$.

Lecture 11

Remark 21. *Let (X, τ) be a topological space, $A, B \subseteq X$. Then*

- $A \subseteq B \Rightarrow Ext(B) \subseteq Ext(A)$.

  Proof. Suppose that $A \subseteq B$, we have $B^c \subseteq A^c$, $(B^c)^0 \subseteq (A^c)^0$ which equivalent to $Ext(B) \subseteq Ext(A)$. □

- $Ext(A \cup B) = Ext(A) \cap Ext(B)$.

  Proof. $Ext(A\cup B) = ((A\cup B)^c)^0 = (A^c \cap B^c)^0 = (A^c)^0 \cap (B^c)^0 = Ext(A) \cap Ext(B)$. □

- $EXT(X) = (X^c)^0 = (\phi)^0 = \phi$.
- $EXT(\phi) = (\phi^c)^0 = (X)^0 = X$.
- $Bd(A) = Bd(X \setminus A)$

  Proof. From definition a point $x \in X$ is a boundary point of A if and only if $\forall$ open set $G : x \in G \Rightarrow G \cap A \neq \phi$, and $G \cap A^c \neq \phi$ and a point $x \in X$ is a boundary point of $X \setminus A$ if and only if $\forall$ open set $G : x \in G \Rightarrow (G \cap X \setminus A) \neq \phi \Rightarrow G \cap A^c \neq \phi$, and $G \cap (X \setminus A)^c \neq \phi \Rightarrow G \cap A \neq \phi$. So, it is clear that they are the same. □

Theorem 10. *Let (x, τ) be a topological space, $A \subseteq X$ is open if and only if A contains none of its boundary points.*

Proof. We start to prove the direction $\Rightarrow$. Suppose that A is open, we need to show $A \cap Bd(A) = \phi$. Since A is open, $A = A^0$ and $A^0 \cap Bd(A) = \phi$, i.e., $A \cap Bd(A) = \phi$.
Now, We prove the direction $\Leftarrow$. Suppose that $A \cap Bd(A) = \phi$, we need to show A is open, i.e., $A = A^0$. It is enough to show $A \subseteq A^0$. Let $x \in A$, then $x \notin Bd(A)$ as $A \cap Bd(A) = \phi$. Also, $A \cap Ext(A) = \phi$ because $x \in A$. But the sets $A^0, Bd(A), Ext(A)$ are disjoint. Thus $x \in A^0$ which completes the proof. □

Corollary 2. *Let (x, τ) be a topological space, $A \subseteq X$ is closed if and only if A contains all of its of its boundary points.*

10 Cluster points, Accumulation points, limit points

Definition 19. *Let (X, τ) be a topological space, $A \subseteq X$, a point $x \in X$ is called a cluster point of A (limit point, accumulation point) if and only if every open set containing x contains points of A other than x, i.e.,*

$$\forall G \in \tau : x \in G, G \cap A \setminus \{x\} \neq \phi.$$

The set of all cluster points of A is called the derived set of A and is denoted by A'.

Remark 22. *Let (X, τ) be a topological space, $A \subseteq X$, a point $x \in X$ is not a limit point of A iff $\exists G \in \tau : x \in G, G \cap A \setminus \{x\} = \phi$.*

Example 41. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}$ be a topology on X. Consider $A = \{a, b, d\}$. Find A'.*

- *Is $a \in A'$? to answer this question we have to test if all the open sets that contain (a) contain other points of A other than (a). If we consider the open set $\{a\}$, we have $a \in \{a\}$ but $A \cap \{a\} \setminus \{a\} = \{a, b, d\} \cap \phi = \phi$. Thus $a \notin A'$.*

- *Is $b \in A'$? to answer this question we have to test if all the open sets that contain (b) contain other points of A other than (b). If we consider the open set $\{b, c\}$, we have $A \cap \{b, c\} \setminus \{b\} = \{a, b, d\} \cap \{c\} = \phi$. Thus $b \notin A'$.*

- *Is $c \in A'$?If we consider the open set X, Its clear that $A \cap X \setminus \{c\} \neq \phi$. Now, $c \in \{b, c\}$ and $A \cap \{b, c\} \setminus \{c\} = \{a, b, d\} \cap \{b\} = \{b\} \neq \phi$,$c \in \{a, b, c\}$ $A \cap \{a, b, c\} \setminus \{c\} = \{a, b, d\} \cap \{a, b\} = \{a, b\} \neq \phi$, $c \in \{b, c, d, e\}$ and $A \cap \{b, c, d, e\} \setminus \{c\} = \{a, b, d\} \cap \{b, d, e\} = \{b, d\} \neq \phi$. So, $\forall G \in \tau : c \in G, G \cap A \setminus \{c\} \neq \phi$. Thus $c \in A'$.*
  Similarly, we can conclude that $d, e \in A'$. Thus, $A' = \{c, d, e\}$.

Example 42. *Consider the topological space $(\mathbb{R}, \tau_{dis})$ and a subset $A \subseteq X$. Describe A'.*
Let $x \in X$, we can find an open set $G = \{x\} : \{x\} \cap A \setminus \{x\} = \phi$. Thus $x \notin A'$ and $A' = \phi$.

Example 43. *Consider the topological space $(\mathbb{R}, \tau_u)$ and the set $A = (0, 1]$, then $A' = [0, 1]$. The solution (in details) is discussed during the lecture.*

Lecture 12

Example 44. *Consider the topological space* $(\mathbb{R}, \tau_u)$ *and the set* $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ *, then* $A' = \{0\}$*. The solution (in details) is discussed during the lecture.*

Example 45. *Consider the topological space* $(\mathbb{R}, \tau_u)$ *and the set* $A = \mathbb{Q}$ *, then* $A' = \mathbb{R}$*. The solution (in details) is discussed during the lecture.*

Example 46. *Consider the topological space* $(\mathbb{R}, \tau_u)$ *and the set* $A = \mathbb{Z}$ *, then* $A' = \phi$*. The solution (in details) is discussed during the lecture.*

Example 47. *Consider the topological space* $(\mathbb{R}, \tau_\ell)$ *and the set* $A = (0, 1] \cup \{2\}$ *, then* $A' = [0, \infty)$*. The solution (in details) is discussed during the lecture.*

Theorem 11. *Let* (x, τ) *be a topological space,* $A \subseteq X$*,* $B \subseteq X$*. Then*

- $\phi' = \phi$.
- *If* $A \subset B$*, then* $A' \subset B'$.

  Proof. Suppose that $A \subseteq B$, we need to show $A' \subset B'$. Let $x \in A'$, then $\forall G \in \tau : x \in G, G \cap A \setminus \{x\} \neq \phi$ but from assumption $A \subset B$, so $G \cap B \setminus \{x\} \neq \phi$. Thus $x \in B'$. □

- $(A \cup B)' = A' \cup B'$.

  Proof. We have $A \subseteq A \cup B$ and $B \subseteq A \cup B, A' \subseteq (A \cup B)'$ and $B' \subseteq (A \cup B)'$. Thus $A' \cup B' \subseteq (A \cup B)'$*
  Now, we need to show $(A \cup B)' \subseteq A' \cup B'$. Let $x \in (A \cup B)'$ and $x \notin A' \cup B'$, i.e., $x \notin A'$ and $x \notin B'$, $\exists G \in \tau : x \in G, G \cap A \setminus \{x\} = \phi$, $\exists H \in \tau : x \in H, H \cap B \setminus \{x\} = \phi$, $\exists G \cap H \in \tau : x \in (G \cap H)$ and $(G \cap H) \cap (A \cup B) \setminus \{x\} = \phi$. This means that $x \notin (A \cup B)'$ which is a contradiction which completes the proof

  □

- $A' \cap B' \subseteq (A \cap B)'$.

Theorem 12. *Let* (x, τ) *be a topological space,* $A \subseteq X$*. Then*
A is closed if and only if A contains all of its cluster points.

Proof. We start to prove the direction $\Rightarrow$. Suppose that A is closed, we need to show $A' \subseteq A$, i.e., if $x \in A'$, then $x \in A$. Assume $x \in A'$ and $x \notin A$, we will seek for a contradiction. Since $x \notin A$, then $x \in A^c$ and A^c is open because A is closed, this means $\exists G \in \tau : x \in G \subseteq A^c$ this implies to $G \cap A = \phi$. Thus $G \cap A \setminus \{x\} = \phi$, i.e., $x \notin A'$ which contradicts our assumption.
Now, we prove the direction $\Leftarrow$. Suppose that $A' \subseteq A$, we need to show A is closed, i.e., A^c is open. Let $x \in A^c$, then $x \notin A$. From assumption we have $A' \subseteq A$, so $x \notin A'$, i.e., $\exists G \in \tau : x \in G, G \cap A \setminus \{x\} = \phi$. As $x \in A^c$ we have $G \cap A = \phi$, i.e., $G \subseteq A^c$. Thus A^c is open and A is closed which completes the proof.

□

Corollary 3. *If F is a closed subset of any set A, then $F' \subset A$.*

Proof. Suppose that $F \subset A$ and F is closed. Then $F' \subset F \subset A$. $\square$

Lecture 13

Theorem 13. *Let (x, τ) be a topological space, $A \subseteq X$, $\overline{A} = A^0 \cup Bd(A)$.*

Theorem 14. *Let (x, τ) be a topological space, $A \subseteq X$, $\overline{A} = A \cup A'$.*

Theorem 15. *Let (X, τ) be a topological space. $A \subset X$. Then $x \in A$ is an isolated point of A if and only if there exists an open G containing x such that $G \cap A = \{x\}$.*

Example 48. *Consider the topological space $(\mathbb{R}, \tau_u)$ and $A = (0, 1] \cap \{2\}$. Find the isolated points of A?*
The only isolated point of A is $\{2\}$.

Example 49. *Consider the topological space $(\mathbb{R}, \tau_u)$ and $A = \mathbb{N}$. Find the isolated points of A?*
The isolated points of A is $\mathbb{N}$.

Remark 23.

- *A given point $x \in X$ belongs to one and only one of the following sets $Ext(A), A'$, and the set of the isolated points of A.*
- *If τ is smaller, then the point $x \in X$ has a better chance to be a cluster point of A.*
- *If τ is larger, then the point $x \in A$ has a better chance to be an isolated point of A.*

Theorem 16. *Let A be a subset of topological space (X, τ). Then the following statements are equivalent*

1. *The set A is dense in X.*
2. *If B is any closed subset of X, and $A \subseteq B$, then $B = X$.*
3. *For each $x \in X$, every open set in X containing x has nonempty intersection with A.*
4. *$(A^C)^0 = \phi$.*

Proof. $1 \Rightarrow 2$, suppose that the set A is dense in X and B is any closed subset of X and $A \subseteq B$, we need to show that $B = X$, It is enough to show $X \subseteq B$. Now, from our assumption $A \subseteq B$, and as we know $\overline{A} \subseteq \overline{B}$. Because A is dense in X, we have $X \subseteq \overline{B}$ but B is closed, i.e., $\overline{B} = B$. Thus $X \subseteq B$.
$2 \Rightarrow 3$, suppose that B is any closed subset of X, and $A \subseteq B$, then $B = X$. Let $x \in X$ and let $U \in \tau$ s.t $x \in U$. We need to show $U \cap A \neq \phi$. Assume the contrary $U \cap A = \phi$, then $A \subseteq U^c$ and U^c is closed, from our assumption we conclude that $U^c = X$, i.e., $U = \phi$ but $x \in U = \phi$ which is a contradiction. Thus $U \cap A \neq \phi$.
$3 \Rightarrow 4$, suppose that $x \in X$, and $\forall U \in \tau, x \in U$, we have $u \cap A \neq \phi$. We need to show $(A^c)^0 = \phi$. Assume the contrary $(A^c)^0 \neq \phi$, then $\exists x \in (A^c)^0$,

i.e., $\exists G \subseteq \tau : x \in G \in A^c$, this implies to $G \cap A = \phi$ which contradicts our assumption. Thus $(A^c)^0 = \phi$.
$4 \Rightarrow 1$, suppose that $(A^C)^0 = \phi$, we need to show $\overline{A} = X$, i.e., $\overline{A} \subseteq X$ and $X \subseteq \overline{A}$. Assume $X \nsubseteq \overline{A}$, i.e., $\exists x \in X$ and $x \notin \overline{A}$, then $x \in (\overline{A})^c$ and $(\overline{A})^c$ is an open set. We know $A \subseteq \overline{A}$, from set theory we have $(\overline{A})^c \subseteq A^c$. The set $(\overline{A})^c$ is an open set contained in A^c and since $(A^c)^0$ is the largest open set contained in (A^c), we conclude that $(\overline{A})^c \subseteq (A^c)^0$ but we have $x \in (\overline{A})^c \subseteq (A^c)^0$, i.e., $(A^c)^0 \neq \phi$ which is a contradiction, thus $X \subseteq \overline{A}$ and this completes the proof. $\square$

Lecture 14

11 Bases and Subbases

Definition 20. *Let (X,τ) be a topological space. A base or basis for τ is a collection β of subsets of X such that:*

1. *Each member of β is also a member of τ.*
2. *If $U \in \tau$ and $U \neq \phi$, then U is the union of sets belonging to β.*

Remark 24. *The elements of β are called basic open sets in X.*

Example 50. *Let $X = \{1,2,3,4\}$ and let $\tau = \{\phi, X, \{1\}, \{2\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2\}\}$ be a topology on X. The set $\beta = \{\{1\},\{2\},\{3,4\}\}$ is a base for τ because all the elements of β are in τ and each element in τ can be written as union of sets belonging to β.*

Example 51. *Consider the topological space $(\mathbb{R}, \tau_{dis})$, the set $\beta = \{\{x\} : x \in \mathbb{R}\}$ is a base for τ_{dis}.*

Example 52. *Let $X = \{a,b,c\}$ and let $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ be a topology on X. Then the only bases for τ are*

- $\beta_1 = \tau$.
- $\beta_2 = \{X, \{a\}, \{b\}\}$.
- $\beta_3 = \{X, \{a\}, \{b\}, \{a,b\}\}$.

Definition 21. *Two collections β_1 and β_2 of subsets of X are equivalent bases if there exists a topology τ on X such that β_1 and β_2 are both bases for τ.*

Remark 25. *$\beta_1, \beta_2, \beta_3$ in Example 52 are equivalent bases for τ.*

Remark 26.

- *Every topology is a base for itself.*
- $\phi = \bigcup\limits_{i=\phi} B_i \in \beta$.
- *There are more than one base for a topology.*
- *A base for a topology need not to be a topology on X.*

Theorem 17. *Let (X,τ)be a topological space and β is a base for τ. Then $U \subseteq X$ is open if and only if $\forall x \in U, \exists B \in \beta$ s.t $x \in B \subseteq U$.*

Proof. We start to prove the direction $\Rightarrow$. Suppose that $U \subseteq X$ is open and let $x \in U$. Because β is a base for τ, we can write U as union of elements of β, i.e., $U = \bigcup\limits_{\substack{\alpha \in \triangle \\ B_\alpha \in \beta}} B_\alpha$. This means there exists $\alpha_0 \in \triangle$ s.t $x \in B_{\alpha_0} \in \beta$ and $B_{\alpha_0} \subseteq \bigcup\limits_{\substack{\alpha \in \triangle \\ B_\alpha \in \beta}} B_\alpha = U$.

Now, we prove the direction $\Leftarrow$. Suppose that $U \subseteq X$ and $\forall x \in U, \exists B \in \beta$ s.t $x \in B \subseteq U$, we need to show U is open. We know that $\beta \subseteq \tau$, i.e., $B \in \tau$. Thus U is open which completes the proof. $\square$

Theorem 18. *For a space (X, τ), a collection $\beta \subseteq \tau$ is a base for τ if and only if $\forall U \in \tau$ and $\forall x \in U\ \exists B \in \beta : x \in B \subseteq U$.*

Example 53. *Consider the topological space $(\mathbb{R}, \tau_u)$ and Let $\beta = \{(a, b) : a, b \in \mathbb{R},\ a < b\}$. Then by Theorem 18 β is a base for the usual topology because $\forall U \in \tau, \forall x \in U, \exists (a, b) \in \beta : x \in (a, b) \subseteq U$.*

11.1 Base of Subspace

Theorem 19. *Let (X, τ) be a topological space and β is a base for τ. If $A \subseteq X$, then $\beta_\alpha = \{B \cap A : B \in \beta\}$ is a base for the subspace topology τ_α on A.*

Proof. The elements of β_α has the form $B \cap A$ and $B \in \beta \subseteq \tau$. Thus $\beta_\alpha \subseteq \tau_\alpha$. Now, Let $U \in \tau_\alpha$ and $x \in U$, $U = G \cap A$ for some $G \in \tau$ and $x \in G \cap A$,i.e., $x \in G$ and $x \in A$,but β is basis for τ so by Theorem 18 $\exists B_0 \in \beta : x \in B_0 \subseteq G$ and $x \in A$, $x \in B_0 \cap A \subseteq G \cap A$. Hence $B_0 \cap A \in \beta_\alpha$, by Theorem 18 we conclude that β_α is a basis for τ_α. $\square$

Example 54. *Let $X = \{a, b, c, d\}$ be a set and $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ be a topology on X. The sub-collection $\beta = \{X, \{a\}, \{b, c\}\}$ form a base for τ. Now, If $A = \{b, c, d\}$ find*

- $\tau_\alpha = \{\phi, A, \{b, c\}\}$.
- $\beta_\alpha = \{A, \{b, c\}, \phi\}$.

Question How we can know if a collection of subsets form a base for some topology? to answer this question, consider the following example

Example 55. *Let $X = \{a, b, c, d\}$ be a set and $\beta = \{\{a\}, \{a, b\}, \{b, c\}\}$is a collection of subsets of X. Can β be a base for some topology. Note that X can not expressed as union of elements of β. Also, as the elements of β are open in τ, then their intersection must be in τ, i.e., $\{b\} \in \tau$ but it can not expressed as a union of elements in β.*

Theorem 20. *Let β be a collection of subsets of nonempty set X. Then β is a base for some topology on X if and only if the following two conditions hold*

1. *$\forall x \in X$, $\exists B \in \beta$ s.th $x \in B$.*
2. *If $B_1, B_2 \in \beta$ and if $x \in B_1 \cap B_2$, then $\exists B_3 \in \beta$ s.th $x \in B_3 \subseteq B_1 \cap B_2$.*

Example 56. *Let $X = \{a, b, c, d\}$ and $\beta = \{\{a, b\}, \{b, c, d\}\}$. Does β a base for a topology on X? It is clear that condition 2 of Theorem 19 doesn't hold since $\{a, b\} \cap \{b, c, d\} = \{b\}$ and there is no $B_3 \in \beta : \{b\} \in B_3 \subseteq B_1 \cap B_2$. So, β is not a base for a topology on X.*

11.2 Exercises

1. State if the following statements are True or False, justify your answer

   - Every topology is a base for itself
   - Let $X = \{1, 2, 3\}$, the set $\beta = \{1, 3\}, \{2, 3\}$ is a base for some topology on X.

Lecture 15

Example 57. *Let $x = \{a, b, c\}$ and $\beta = \{\{a\}, \{b\}, \{c\}\}$.*

- *Show that β is a base for some topology on X. We need to verify conditions (1) (2) of Theorem 19 (do it as an exercise).*
- *Find $\tau(\beta) = \{\{a\}, \{b\}, \{c\}, \phi, X\}$.*

Theorem 21. *Let (X, τ_1), (X, τ_2) be topological spaces with bases β_1, β_2 respectively. Then*

$$\tau_1 \subseteq \tau_2 \Leftrightarrow \forall B_1 \in \beta_1, x \in \beta_1, \exists B_2 \in \beta_2 : x \in B_2 \subseteq B_1$$

Proof. $\Rightarrow$ Suppose that $\tau_1 \subseteq \tau_2$, let $B_1 \in \beta_1, x \in B_1$, $B_1 \in \tau_1$, since $\tau_1 \subseteq \tau_2$ $B_1 \in \tau_2$,by Theorem 18 we have $\forall x \in B_1 : x \in B_1, \exists B_2 \in \beta_2 : x \in B_2 \subseteq B_1$. Conversely, $\Leftarrow$ suppose that $\forall B_1 \in \beta_1, x \in B, \exists B_2 \in \beta_2 : x \in B_2 \subseteq B_1$, let $U \in \tau_1$, then $\exists B_1 \in \beta_1 : x \in B_1 \subseteq U$, from assumption $\exists B_2 \in \beta_2 : x \in B_2 \subseteq B_1 \subseteq U$. Thus U is open in τ_2 which completes the proof. □

12 Finite product of Topological Space

Recall that

- If $X_1 \times X_2 \times \cdots \times X_n$ are sets, then the Cartesian product is the set of all n- tuples of $X_1 \times X_2 \times \ldots X_n = \{(x_1, x_2, \ldots, x_n) : x_i \in X_i\}$.
- If $A \subset C, B \subset D$, then $A \times B \subseteq C \times D$.
- $X \times \phi = \phi \times X = \phi$.
- $A \times B = \phi$ if and only if $A = \phi$ or $B = \phi$.
- $X \times (Y \cap Z) = (X \times Y) \cap (X \times Z)$.
- $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.
- $(X \times Y) \cap (Z \cap W) = (X \cap Z) \times (Y \cap W)$.
- $(X \times Y) \cup (Z \cap W) \subseteq (X \cup Z) \times (Y \cup W)$.

Now, if τ_i is a topology on X_i for $i = 1, 2, \ldots, n$, we will topologize the product set $X_1 \times X_2 \times \cdots \times X_n$.

Theorem 22. *Let (x_i, τ_i), $i = 1, 2, \ldots, n$ be a finite collection of topological spaces and let $X = X_1, X_2, \ldots, X_n$ be the set of Cartesian product. If β is the collection of all sets of X of the form $U_1 \times U_2 \times \ldots U_n$ where $U_i \in \tau_i$, then β is a base for a topology on X.*

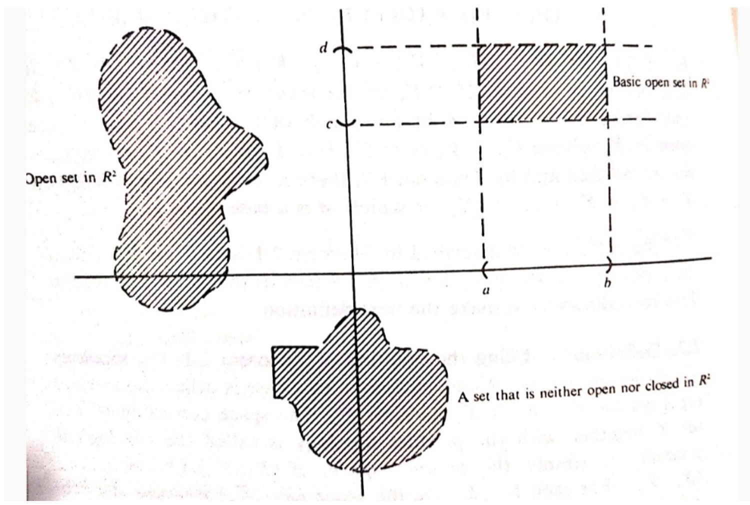
Definition 22. *The topology $\tau(\beta)$ generated by β and having β as a base is called the product topology on $X = X_1 \times X_2 \times \cdots \times X_n$. The space consisting of the set X together with the product topology is called the topological product.*

Remark 27.

- *If (X, τ_p) is a product space of spaces (X_k, τ_k), $k = 1, 2, \ldots, n$, then $u \in \tau_p$ if and only if $\forall x \in u, \exists B \in \beta : x \in B \subseteq u$.*

- *In the plane $\mathbb{R}^2$, an open rectangle is a set of the form $(a, b) \times (c, d)$. Since (a, b) and (c, d) are open in $\mathbb{R}$, the open rectangles belong to the base β of $\mathbb{R}^2$.*

- *In the plane $\mathbb{R}^2$, the open sets are not necessary only open rectangles as shown in the figure below.*



Lecture 16

13 Continuous Function

Definition 23. *Let f be a function from $\mathbb{R} \to \mathbb{R}$. The function f is continuous at $x = a \in \mathbb{R}$ if for any $\epsilon > 0$, there exists a real number $\delta > 0$ s.t*

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon,$$

this is equivalent to

$$x \in (-\delta + a, \delta + a) \Rightarrow f(c) \in (f(a) - \epsilon, f(a) + \epsilon),$$

$$f(-\delta + a, \delta + a) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$$

Definition 24. *Let (X, τ_x) and (Y, τ_y) be topological spaces and the function $f : X \to Y$. The function f is continuous at the point $x_0 \in X$ if and only if for any open set V containing $f(x_0)$, there exists an open set $U \subset X$ containing x_0 such that $f(U) \subseteq V$, i.e.,*

$$\forall V \in \tau_y : f(x_0) \in V, \exists U \in \tau_x \;\; s.t \;\; f(U) \subseteq V, U \subseteq f^{-1}(V).$$

Remark 28.
- *The Definition 24 agree with epsilon-delta definition.*
- *The smaller topology on Y and the larger topology on X, the better chance for the function f to be continuous.*
- *A given function may be continuous for certain topologies on X and Y but not continuous if different topologies are defined on theses sets.*

Example 58. *Let $f : X \to Y$ be a function given by $f(x) = x$, $X, Y = \mathbb{R}$ with usual topology. Show that f is continuous at each $x_0 \in X$.*
Let $x_0 \in X$ and $V \in \tau_y : f(x_0) \in V$, $f(x_0) = x_0 \in V$, choose $U = V$ and $f(U) = U \subseteq V$. Thus f is continuous at each $x_0 \in X$.

Example 59. *Let $f : X \to Y$ be a function given by $f(x) = x$, $X, Y = \mathbb{R}$ with cofinite and usual topology respectively. Show that f is discontinuous at $x_0 = 1$.*
We have $f(1) = 1$, there exists $V = (0, 2) \in \tau_y : f(1) = 1 \in V$. Any U open in τ_x and $1 \in U$ has the form $\mathbb{R} \setminus \{finite\, set\}$ and $f(U) = U$. Finally $f(U) \nsubseteq V$. Thus f is discontinuous at $x = 1$.

Discontinuity Criterion One The function $f : (X, \tau_x) \to (Y, \tau_y)$ is discontinuous at $x = x_0$ if and only if $\exists V \in \tau_y : f(x_0) \in V$ and $\forall U \in \tau_x, x_0 \in U, \; f(U) \nsubseteq V$.

Example 60. *Let $f : X \to Y$ be a function given by $f(x) = x$, $X, Y = \mathbb{R}$ with usual and discrete topology respectively. Show that f is discontinuous at any $x_0 \in X$.*
We have $f(x_0) = x_0$, there exists $V = \{x_0\} \in \tau_{dis} : f(x_0) = x_0 \in V$. Any U open in τ_x and $x_0 \in U$ has the form $(x_0 - \epsilon, x_0 + \epsilon)$ and $f(U) = U$. Finally $f(U) = (x_0 - \epsilon, x_0 + \epsilon) \nsubseteq V = \{x_0\}$. Thus f is discontinuous at $x = x_0$.

Example 61. *Let $f : X \to Y$ be a function given by $f(x) = x$, $X, Y = \mathbb{R}$ with discrete and usual topology respectively. Show that f is continuous at any $x_0 \in X$.*
Let $x_0 \in X$ and $V \in \tau_y : f(x_0) \in V$, $f(x_0) = x_0 \in V$, choose $U = \{x_0\} \in \tau_x$ and $x_0 \in U$ and $f(U) = U = \{x_0\} \subseteq V$. Thus f is continuous at each $x_0 \in X$.

Theorem 23. *Let $f : X \to Y$ be a function from one topological space to another. Then the following conditions are equivalent.*

1. *The function f is continuous.*
2. *For each open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.*
3. *For each closed set $M \subseteq Y$, $f^{-1}(M)$ is closed in X.*

Proof. First, we will prove the direction $1 \Rightarrow 2$. Suppose that the function f is continuous, and let $V \subseteq Y$ be an open set. We need to show $f^{-1}(V)$ is open in τ_x, i.e., $\forall x \in f^{-1}(V), \exists H \in \tau_x : x \in H \subseteq f^{-1}(V)$. Let $x \in f^{-1}(V)$, this means $f(x) \in V$. Because f is continuous there exists $U \in \tau_x$ such that $x \in U$ and $f(U) \subseteq V$, $U \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is an open set.
Now, we will prove the direction $2 \Rightarrow 1$. Suppose that for each open set $V \subseteq Y$, $f^{-1}(V)$ is open in X. We need to show f is continuous. Let V be any open set in Y such that $f(x) \in V$, from assumption $f^{-1}(V)$ is open in X and $x \in f^{-1}(V)$. Choose $U = f^{-1}(V)$, then $x \in U = f^{-1}(V)$ and $f(U) = f(f^{-1}(V)) \subseteq V$. Thus f is continuous. □

Example 62. *Let $X = \{a, b, c\}$ and we define $\tau_x = \{\phi, X, \{a\}, \{b, c\}\}$ to be a topology on X. Let $Y = \{x, y, z\}$ and we define $\tau_y = \{\phi, Y, \{y\}, \{z\}\{y, z\}\}$ to be a topology on Y. Prove or disprove if the function $f = \{(a, x), (b, z), (c, z)\}$ is continuous.*
We want to show if the pre image of any open set in Y is open in X.
$f^{-1}(\phi) = \phi$ which is open in X.
$f^{-1}(Y) = X$ which is open in X.
$f^{-1}(\{y\}) = \phi$ which is open in X.
$f^{-1}(\{z\}) = \{b, c\}$ which is open in X.
$f^{-1}(\{y, z\}) = \{b, c\}$ which is open in X.
Thus f is continuous.

Example 63. *Let $f : (X, \tau_{dis}) \to (Y, \tau)$ be a function where τ is any topology on Y. Prove that f is continuous.*
Let V be any open set in τ, $f^{-1}(V) \subseteq X$, $f^{-1}(V) \subseteq \mathcal{P}(x)$, $f^{-1}(V) \in \tau_{dis}$. Thus f is continuous.

Example 64. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function where $f(x) = a$ for all $x \in X$. Prove that f is continuous.*
Let V be any open set in τ, then

$$f^{-1}(V) = \begin{cases} X & \text{if } a \in V \\ \phi & \text{if } a \notin V \end{cases}.$$

In both cases $f^{-1}(V)$ is open in τ_1.

Discontinuity Criterion Two The function $f : (X, \tau_1) \to (Y, \tau_2)$ is discontinuous if and only if there is some open set V open in τ_2 s.t $f^{-1}(V)$ is not open in X.

Example 65. *Let $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$*
$f(x) = x^2$.

1. *Find the inverse inverse image of the following set*
   - $[0, \infty) = \mathbb{R} \setminus \{0\}$
   - $(0, \infty) = \mathbb{R}$
   - $(-3, \infty) = \mathbb{R}$
   - $(4, \infty) = (-\infty, 2) \cup ()2, \infty)$
   - $[0, 4) = (-2, 2)$

2. *Show that f is continuous on $\mathbb{R}$.*
   Let $V = (a, b)$ be any open interval in $Y = \mathbb{R}$ containing $f(x)$, then

$$f^{-1}(V) = \begin{cases} \phi & \text{if } a < 0, b < 0 \\ (-\sqrt{b}, \sqrt{b}) & \text{if } a < 0, b > 0 \\ (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b}) & \text{if } a > 0, b > 0 \end{cases}.$$

   In all cases $f^{-1}(V)$ is open in τ_u. Thus f is continuous on $\mathbb{R}$,

Example 66. *Let $f : (X, \tau_u) \to (Y, \tau_u)$ be a function where $X = [0, 7], Y = \mathbb{R}$, and*

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 4 \\ 15 & \text{if } 4 < x \leq 7 \end{cases}.$$

Show that f is not continuous at $x = 4$.
Let $V = (11, 13)$ be an open set in Y such that $f(4) = 12 \in V$, for any U open set in X containing 4, then U contains points $x > 4$ and $f(U) = 15$, $f(U) \nsubseteq (11, 13)$.

Example 67. *Let $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_\ell)$ be a function where $f(x) = |x|$, show that f is continuous.*
Let V be any open set in $(\mathbb{R}, \tau_\ell)$, then V has the one of the following forms $\phi, \mathbb{R}, (-\infty, r)$. Now $f^{-1}(\phi) = \phi, f^{-1}(\mathbb{R}) = \mathbb{R}, f^{-1}(-\infty, r) = (-r, r)$. In all cases $f^{-1}(v)$ is open in τ_u.

Example 68. *Let $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_r)$ be a function where $f(x) = |x|$, show that f is continuous.*
***HW**.*

13.1 Composition of Continuous function

Theorem 24. *If $f : X \to Y$ and $g : Y \to Z$ are both continuous functions, then the composition $g \circ f : X \to Z$ is continuous.*

Proof. Let V be an open set in Z, $g^{-1}(V)$ is open set in Y since g is continuous, $f^{-1}(g^{-1}(V))$ is open set in X since f is continuous. Thus $(g \circ f)^{-1}(V)$ is an open set in X, i.e., $g \circ f$ is continuous. □

13.2 Exercises

1. State two conditions on a collection of subsets of a nonempty set X to be a base for some topology on X.
2. Define the product topology on the topological spaces $(X_1, \tau_1), (X_2, \tau_2), \ldots, (X_n, \tau_n)$.
3. Let $f : (X, \tau_x) \to (Y, \tau_y)$ be a function. Give three equivalent conditions such that the function f to be continuous.
4. Let $f : (X, \tau_x) \to (Y, \tau_y)$ be a function. Give two equivalent conditions such that the function f to be discontinuous.

14 Open Functions and Homomorphisim

14.1 Open and Closed Functions

Definition 25. *Let (X, τ_x) and (Y, τ_y) be a topological spaces and let $f : X \to Y$ be function. Then*

- *f is open function if and only for any open set in X, $f(G)$ is open in Y.*
- *f is closed function if and only for any closed set in X, $f(G)$ is closed in Y.*

Remark 29. *The function f may be continuous and not open, or continuous and open but not closed, or continuous, open and closed.*

Example 69. *Let $X = \{a, b, c\}$ and $\tau_x = \{\phi, X, \{a\}, \{b, c\}\}$ be a topology defined on X. $X = \{x, y, z\}$ and $\tau_y = \{\phi, Y, \{y\}\}$ be a topology defined on Y. Define the function $f : X \to Y$ as follows: $f(a) = x, f(b) = y, f(c) = y$.*

- *Show that f is continuous.*
  $f^{-1}(\phi) = \phi, f^{-1}(Y) = X, f^{-1}(\{y\}) = \{b, c\}$so, thr inverse of any open set in Y is open in X.
- *Show that f is not open.*
  $f(\{a\}) = \{x\}$so, there exists an open set in X such that its image is not open in Y.

- *Show that f is not closed.*
  $f(\{a\}) = \{x\}$*so, there exists a closed set in X such that its image is not closed in Y.*

Example 70. *Let $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_{dis})$ be a function such that $f(x) = 5$. We can show that f is open, closed and continuous function (verify as HW).*

Theorem 25. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function. Then*

$$f \text{ is open } \Leftrightarrow f(A^0) \subseteq (f(A))^0, \, \forall A \subseteq X$$

Proof. $\Rightarrow$ Suppose that f is open and let $A \subseteq X$, $A^0 \subseteq A$, but A^0 is an open set and f is open, so $f(A^0)$ is open in Y and $f(A^0) \subseteq f(A), (f(A^0))^0 \subseteq (f(A))^0$ since $(f(A^0))^0 = f(A^0)$, this equivalent to $(f(A^0) \subseteq (f(A))^0$.
Conversely $\Leftarrow$ Suppose that $(f(A^0) \subseteq (f(A))^0$, we want to show f is open. Let G be any open in X, then $G = G^0$. $f(G) = f(G^0) \subseteq (f(G))^0 \subseteq f(G)$, $f(G) = (f(G))^0$, i.e., $f(G)$ is open in Y and thus F is an open function which completes the proof. □

14.2 Exercises

(a) Give an example for a function f which is continuous, closed but not open.

(b) Let $f : X \to Y$, $g : Y \to Z$ be two functions. Prove that if f and g are open, then $g \circ f$ is open.

Lecture 17

14.3 Homeomorphism

Definition 26. *Let $f : X \to Y$ be a bijection function from the space X to the space Y. If f is open and continuous, then f is called a homeomorphism. If f is a homeomorphism from X to Y, then the space X and Y are said to be homeomorphic denoted by $X \cong Y$.*

Example 71. *Let $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_{dis})$ be a function such that $f(x) = 5$. Then f is not homeomorphism, since it is not onto.*

Theorem 26. *Let $f : X \to Y$ be bijective. Then the following are equivalent:*

1. *f is homeomorphism.*
2. *f and $f^{-1} : Y \to X$ are both continuous.*
3. *f is continuous and closed.*

Proof. $1 \Rightarrow 2$, suppose that $f : X \to Y$ is a bijective and homeomorphism. It cleae that f is continuous, we need to show that it is open. Let V be an open set in X, then $(f^{-1})^{-1}(V) = f(V)$ which is open in Y because f is an open function.
$2 \Rightarrow 3$, suppose that f and $f^{-1} : Y \to X$ are both continuous, we need to show f is closed. Let U be a closed set in X, $(f^{-1})^{-1}(U) = f(U)$ is closed in Y which completes the proof. $2 \Rightarrow 3$.
do it as exercise. □

Definition 27. *A property of a space X is called a topological property if and only if every space Y homemorphic to X also has the same property.*

15 Separation Axioms

Definition 28. *Let (X, τ) be a topological space.*

1. *A space X is a T_0 space if and only if for any $x \neq y$ in X, there exists an open set G such that $x \in G$ and $y \notin G$ or there exists open set H such that $y \in H$ or $x \notin H$.*
2. *A space X is a T_1 space if and only if for any $x \neq y$ in X, there exists an open set G such that $x \in G$ and $y \notin G$ and there exists open set H such that $y \in H$ or $x \notin H$.*
3. *A space X is a T_2 space if and only if for any $x \neq y$ in X, there exist two disjoint open sets G, H such that $x \in G$ and $y \in H$.*

Remark 30.
- *If X is a T_2 space, then sometines X is called Hausdorf Space.*

- *The implication $T_2 \Rightarrow T_1 \Rightarrow T_0$ is true, the converse is not true as we will see in the following examples.*

Example 72. *Let $X = \{a, b\}$ and $\tau = \{\phi, X, \{a\}\}$ be a topology on X.*

- *Show that X is T_0 space.*
  We have $a \neq b$, we need to show there exists an open set G such that $a \in G$ and $b \notin G$. Choose $G = \{a\}$.

- *Show that X is not T_1 space.*
  The only open set that contains b is X and it contains a, so X is not T_1 space.

Example 73. *Consider the topological space $(\mathbb{R}, \tau_\ell)$.*

- *Show that $(\mathbb{R}, \tau_\ell)$ is a T_0 space.*
  Let $x, y \in \mathbb{R}$ and $x \neq y$ we need to show there exists an open set G such that $x \in G$ and $y \notin G$. Suppose $x < y$, let $r = \frac{y-x}{2}$. Choose $G = (-\infty, x + r)$, then $x \in G$ and $y \notin G$. Thus $(\mathbb{R}, \tau_\ell)$ is a T_0 space.

- *Show that X is not T_1 space.*
  Let $x, y \in \mathbb{R}$ and $x \neq y$ Let G be an open set that contains y, then $G = (-\infty, k)$ $k > y$ or $G = \mathbb{R}$. In both cases $x \in G$, so $(\mathbb{R}, \tau_\ell)$ is not T_1 space.

Remark 31. *Example 73 shows that the implication $T_0 \Rightarrow T_1$ is not true.*

Example 74. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$ be a topology on X. Show that X is not T_0 space.*
We have $b \neq c$, the only open open set that contains b is X and it contains c. Also, the only open open set that contains c is X and it contains b. Thus X is not T_0 space.

Example 75. *Consider the topological space $(\mathbb{R}, \tau_{cof})$.*

- *Show that $(\mathbb{R}, \tau_{cof})$ is a T_1 space.*
  Let $x, y \in \mathbb{R}$ and $x \neq y$ we need to show there exists an open set G such that $x \in G$ and $y \notin G$ and an open set H such that $y \in H$ and $x \notin H$, let $G = \mathbb{R} \setminus \{y\}$ and $H = \mathbb{R} \setminus \{x\}$. It is clear that $x \in G$ and $y \notin G$, $y \in H$ and $x \notin H$, . Thus $(\mathbb{R}, \tau_{cof})$ is a T_1 space.

- *Show that $(\mathbb{R}, \tau_{cof})$ is not a T_2 space.*
  For any open sets G such that $x \in G$ and $y \notin G$ and H such that $y \in H$ and $x \notin H$, $G \cap H \neq \phi$. Thus $(\mathbb{R}, \tau_{cof})$ is not a T_2 space.

Example 76. *Show that $(\mathbb{R}, \tau_u)$ is a T_2 space.*
Let $x, y \in \mathbb{R}$ and $x \neq y$, let $r = \frac{y-x}{2}$, consider the open sets $G = (x - r, x + r)$, $H = (y - r, y + r)$, it is clear that $G \cap H = \phi$ and $x \in G$ and $y \notin G$, $y \in H$ and $x \notin H$. Thus $(\mathbb{R}, \tau_u)$ is a T_2 space.

15.1 Some Properties of T_1 Space

Theorem 27. *X is T_1 space if and only if every singlton set is closed in X.*

Proof. Suppose that X is T_1 space, we want to prove that foe any $x \in X$, $\{x\}$ is closed. To prove $\{x\}$ is closed, we need to show $X \setminus \{x\}$ is open, i.e., $\forall y \in X \setminus \{x\}, \exists G \in \tau : y \in G \subseteq X \setminus \{x\}$. Let $y \in X \setminus \{x\}$, then $x \neq y$. Since X is T_1 space, there exists two open sets G, H s.t $x \in G$ and $y \notin G$, $y \in H$ and $x \notin H$ and $G \subseteq X \setminus \{x\}$. Thus, the set $X \setminus \{x\}$is open in X, i.e., $\{x\}$ is closed in X.
Conversely, Suppose that $\{x\}$ is closed in X, we want to show X is T_1 space. Let $x, y \in X$ and $x \neq y$, then $\{x\}, \{y\}$ are closed in X. Let $G = X \setminus \{x\}$, $H = X \setminus \{y\}$ are open in X and $y \in G$ and $x \notin G$, $x \in H$ and $y \notin H$. Thus X is T_1 space which completes the proof. $\square$

Corollary 4. *Let X be a T_1 space then every finite set is closed in X.*

Proof. Let $A = \{x_1, x_2, \ldots, x_n\}$ be a finite set, now by Theorem 27 we conclude that $A = \{x_1\} \cup \{x_2\} \cup \ldots \{x_n\}$ is closed since it is a finite union of closed sets. $\square$

Theorem 28. *The topological space is (X, τ) is a T_1 space if and only if τ contains the cofinite topology on X, i.e., $\tau_c of \subseteq \tau$.*

Proof. Suppose that (X, τ) is a T_1, Let $G \in \tau_{cof}$, $G = X \setminus \{\text{finite set}\}$, $G^c = \{\text{finite set}\}$, since (X, τ) is a T_1 and by Theorem 27 we conclude G^c is closed set (X, τ), i.e., G is open in (X, τ), $G \in \tau$.
Conversely, suppose that $\tau_{cof} \subseteq \tau$, let $x, y \in X$ and $x \neq y$, let $G = X \setminus \{x\}$, $H = X \setminus \{y\}$ are open in $\tau_{cof} \subseteq \tau$, it is clear that $y \in G$ and $x \notin G$, $x \in H$ and $y \notin H$. Thus (X, τ) is a T_1 space. $\square$

Theorem 29. *Every subspace of T_1 space is T_1 space.*

Proof. Suppose that (X, τ) is a T_1 space. For any $A \subseteq X$, define τ_α on A to be $\tau_\alpha = \{A \cap G : G \in \tau\}$, we need to show (A, τ_α) is T_1 space. let $x, y \in A$ and $x \neq y$, then $x, y \in X$ and X is T_1 space, then $\exists G, H \in \tau : x \in G$ and$y \not{\!\!\;} in G$ and $x \notin H, y \in H$, $x \in G \cap A, y \notin G \cap A$ and $x \notin H \cap A, y \in G \cap A$. Since $G \cap A \in \tau_\alpha$, we conclude that every subspace of T_1 space is T_1 space which completes the proof. $\square$

Remark 32.

- *Every subspace of T_0 space is T_0 space.*
- *Every subspace of T_2 space is T_2 space.(try to prove)*

15.2 Topological Property

Definition 29. *A property of a space is called a topological property if and only if every space Y homeomorphic to X has the same property.*

Example 77. *Show that T_0-property is a topological property.*

Proof. Let $f : X \to Y$ be a homeomorphism and suppose that X is a T_0-space. We need to show Y is T_0 space. Let $y_1, y_2 \in Y, y_1 \neq y_2$, since f is one to one and onto $\exists x_1, x_2 \in X : x_1 \neq x_2$ and $f(x_1) = y_1$, $f(x_2) = y_2$. Because X is T_0 space, $\exists$ open set $G : x_1 \in G$ and $x_2 \notin G$ or $\exists$ open set $H : x_1 \notin H$ and $x_2 \in H$. $f(G), f(H)$ are open in Y because f is an open function and $f(x_1) \in f(G), f(x_2) \notin f(G)$ or $f(x_1) \notin f(H), f(x_2) \in f(H)$ i.e., $y_1 \in f(G), y_2 \notin f(G)$ or $y_1 \notin f(H), y_2 \in f(H)$. This means that there an open set in Y containing y_1 but not y_2 and another open set in Y containing y_2 but not y_1. Thus Y is T_0 space. □

Remark 33. *In the same way as example 77, we can show that T_1, T_2 are topological properties.*

Example 78. *Let (X, τ)be a Hausdorff space. If τ_1 is a topology for X such that $\tau \subseteq \tau_1$, then prove that (X, τ_1) is also Hausdorff.*

Proof. Let (X, τ) be a Hausdorff space and $\tau \subseteq \tau_1$. Let $x, y \in X$ and $x \neq y$. Since X is T_2- space $\exists G \in \tau : x \in G$ and $y \notin G$ and $\exists H \in \tau : x \notin H$ and $y \in H$ and $G \cup H = \phi$. Since $\tau \subseteq \tau_1$, we have $G, H \in \tau_1$, i.e., (X, τ_1) is a Hausdorff space. □

15.3 Normal and Regular Spaces

Definition 30. *The space X is called a regular space if and only if for each closed subset $F \subseteq X$ and for each $x \notin F$, there exists two disjoint open sets U and V such that $x \in U$ and $F \subset V$. A regular T_1- space is called a T_3- space.*

Definition 31. *The space X is called a normal space if and only if for each pair of disjoint closed subsets F_1 and F_2 of X, there exists two disjoint open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V$. A normal T_1- space is called a T_4- space.*

Example 79. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$.*

- *Show X is normal.*
  The closed sets are $\{\phi, X, \{b, c\}, \{c\}\}$. Since there is no disjoint closed sets, X is normal.

- *Show X is not T_2.*
  Take $b \neq c$, the only set contains c is X and intersects with any open set contains b.

- *Show that X is not regular.*
  Consider the closed set $\{b,c\}$ and $a \notin \{b,c\}$, the only closed set contains a is X and it intersects with the set $\{b,c\}$.

Theorem 30. *Every T_3 space is T_2- space.*

Proof. Let $x, y \in X$, $x \neq y$. since X is T_1- space, the set $\{y\}$ is closed in X. Clearly $x \notin \{y\}$, because X is regular space there exists disjoint open sets U and $V : x \in U, \{y\} \subseteq V$, $x \in U, y \in V$. Thus X is T_2-space. □

Theorem 31. *Every T_4 space is T_3- space.*

Proof. Let $x \in X$, F is closed set, $x \notin F$. since X is T_1- space, the set $\{x\}$ is closed in X. Clearly the sets $F, \{x\}$ are disjoint closed set. Because X is normal space there exists disjoint open sets U and $V : \{x\} \subseteq U, F \subseteq V$, $x \in U, F \subseteq V$. Thus X is a regular T_1-space, i.e., X is T_3-space. □

Remark 34. *The implication $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ is true.*

15.4 First Axiom of Countability

Definition 32. *Let (X, τ) be a topological space and $x \in X$. A family β_x of open sets which contains x is called a local base at x if $\forall u \in \tau : x \in U, \exists V \in \beta_x : x \in V \subseteq U$.*

Example 80. *Let (X, τ) be a topological space and $x \in X$. The collection $\beta_x = \{V \in \tau : x \in V\}$ is a local base at x.(try to verify)*

Example 81. *Let $X = \{a,b,c,d\}$, and let $\tau = \{\phi, X, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ be a topology on X. Then*

- *$\beta_a = \{X, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ is a local base at a. (Verify).*
- *$\beta'_a = \{\{a\}\}$ is a local base at a.*
- *$\beta_b = \{X, \{a,b\}, \{a,b,c\}\}$ is a local base at b.*
- *$\beta'_b = \{\{a,b\}\}$ is a local base at b.*
- *Try to find two local bases for c, d.*

Definition 33. ***First Axiom of countability*** *A space X is called first countable if $\forall x \in X$, there exists a countable local base at x.*

Example 82. *The following are examples of first countable space.*

- *Any finite topological space.*
- *Any finite topology on any set.*
- *Any discrete space, consider the set $\beta_x = \{\{x\}\}$ is a countable local base. (verify).*

- *The space $(\mathbb{R}, \tau_u)$. To verify this space is a first countable space consider the set $\beta_x = \{(x-r, x+r) : r \in \mathbb{Q}^+\}$ is a countable local base. Clearly the elements of β_x are open sets and the cardinality of β_x is the same as the cardinality of $\mathbb{Q}^+$. Now let $U \in \tau$ and $x \in U$ from the definition of open sets in τ_u, there exists $(a, b) : x \in (a, b) \subseteq U$. Let $\epsilon = min\{|x-a|, |x-b|\}$. There exists $r \in \mathbb{Q}^+$ such that $0 < r < \epsilon$ and $x \in (x-r, x+r) \subseteq (a, b)$. Thus the set $\beta_x = \{(x-r, x+r) : r \in \mathbb{Q}^+\}$ is a countable local base, i.e., the space $(\mathbb{R}, \tau_u)$ is first countable space.*

Theorem 32. *Let X be a first countable space and let $x \in X$. Then there exists a countable local base $\{U_n : n \in \mathbb{N}\}$ at x such that*

1. *$U_{n+1} \subseteq U_n, \forall n \in \mathbb{N}$.*
2. *If X is also a T_1-space, then $\bigcap\limits_{n\in\mathbb{N}} = \{x\}$.*

Proof. Suppose that X is a first countable space. For all $x \in X$, X has a countable local base $\beta_x = \{v_1, v_2, \dots, v_n, \dots\}$, i.e., $\forall G \in \tau : x \in G, \exists v \in \beta_x : x \in v \subseteq G$. Now, Let $U_1 = v_1, U_2 = v_1 \cap v_2, U_3 = v_1 \cap v_2 \cap v_3, \dots$, we will show that the set $\{U_1, U_2, \dots\}$ is countable local base and $U_{n+1} \subseteq U_n, \forall n \in \mathbb{N}$.

- $x \in v_i \Rightarrow x \in U_i$.
- $U_i,\ i \in \mathbb{N}$ is an open set since its intersection of open sets. Note that $U_1 \subseteq v_1$, $U_2 \subseteq v_2, \dots$
- Let $x \in X$, since β_x is a local base, we have $\forall H \in \tau, \exists v_i \in \beta_x : x \in v_i \subseteq H$. So, $x \in U_i \subseteq v_i \subseteq H$.

Thus, the set $\{U_n : n \in \mathbb{N}\}$ is local base and a countable set. Also, $U_2 \subseteq U_1$, $U_3 \subseteq U_2, \dots$ which completes the proof of 1.
Now to prove 2, Suppose that X is a T_1 space.
Claim $\bigcap\limits_{n\in\mathbb{N}}^{\infty} U_n = \{x\}$.
We proved in 1 that the collection $\{U_n : n \in \mathbb{N}\}$ is a local base, i.e., $\forall x \in X, x \in U_i, \forall i \in \mathbb{N}$ so, $x \in \bigcap\limits_{i=1}^{\infty} U_i$, $\{x\} \subseteq \bigcap\limits_{i=1}^{\infty} U_i$, we need to show $\bigcap\limits_{i=1}^{\infty} U_i \subseteq \{x\}$ by contrapositive. Let $y \notin \{x\}$, $y \neq x$. Since X is T_1 space, $\exists G, H \in \tau$ s.th $x \in G$ and $y \notin G$, $x \notin H, y \in H$. Because the set $\{U_n : n \in \mathbb{N}\}$ is a countable local base we have for some $m \in \mathbb{N}$ $\exists U_m \in \{U_n : n \in \mathbb{N}\}$ s.th $x \in U_m \subseteq G$ because $y \notin G$, this imply to $y \notin U_m$ for some $m \in \mathbb{N}$, $y \notin \bigcap\limits_{i=1}^{\infty} U_i$. This proves that $\bigcap\limits_{i=1}^{\infty} U_i \subseteq \{x\}$ which completes the proof.

□

15.5 Second Axiom of Countability

Definition 34. *A space (X, τ) is called second countable (or satisfies the second axiom of countability) if and only if there is a countable base for τ.*

Example 83. *Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ be a topology on X. Then $\beta_a = \{\{a\}\}$ is a local base but not a base.*

Theorem 33. *Every second countable space is a first countable space.*

Example 84.

1. *Any finite topological space is second countable.*
2. *Any finite topology on any set is second countable.*
3. *Any discrete space is second countable if and only if X is countable.*
4. *The space $(\mathbb{R}, \tau_u)$ is second countable. To verify this consider $\beta = \{(r, s) : r, s \in \mathbb{Q}, \}$, $|\beta| = |\mathbb{Q} \times \mathbb{Q}|$ so, it is countable and contains open sets. Let $u \in \tau, x \in u$, then $\exists (a, b) : x \in (a, b) \subseteq u$. We know that between any two real numbers, there exists a rational number, thus $\exists (r, s) : r, s \in \mathbb{Q}$, $x \in (r, s) \subseteq (a, b) \subseteq u$. Hence β is a countable base.*

Remark 35. *Every second countable space is a first countable space but the converse is not true, consider $(\mathbb{R}, \tau_{dis})$ is a first countable but not a second countable space.*

Theorem 34. *Any subspace of second countable space is second countable.*

Theorem 35. *The property of being second countable is a topological property.*

Definition 35. *A space X is called separable if and only if there exists a countable subset of X which is dense in X.*

Example 85. *The space $(\mathbb{R}, \tau_u)$ is a separable space, the countable dense subset is $\mathbb{Q}$.*

Theorem 36. *Every second countable space is separable.*

Proof. Let X be a second countable space, then there exists a countable base $\beta = \{B_1, B_2, \ldots\}$, pick $x_i \in B_i, i \in \mathbb{N}$
claim the set $D = \{x_i : i \in \mathbb{N}\}$ is a countable dense set. It is clear that $|D| \leq |\beta|$. To show D is dense, it is enough to show that $\forall u \in \tau : x \in u, u \cap A \neq \phi$, see Theorem (15-3). Let u be an open set s.th $x \in u$, $\exists B_m \in \beta : x_m \in B_m \subseteq u, m \in \mathbb{N}$, $x_m \in D$, $u \cap D \neq \phi$. Thus D is a countable dense set which completes the proof. □

16 compact Space

Definition 36. *Consider the topological space (X, τ). The set $G = \{G_\alpha : \alpha \in \triangle\} \subseteq \tau$ is an open cover of X if $X = \bigcup_{\alpha \in \triangle} G_\alpha$.*

Example 86. *Consider the space $(\mathbb{R}, \tau_u)$. The following are open covers of $\mathbb{R}$*

- *$G = \{(-n, n) : n \in \mathbb{N}\}$ since $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$.*

- $H = \{(-\infty, n) : n \in \mathbb{N}\}$ *since* $\mathbb{R} = \bigcup_{n\in\mathbb{N}} (-\infty, n)$.
- $K = \{(n, n+2) : n \in \mathbb{Z}\}$ *since* $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} (n, n+2)$.

Definition 37. *Let $A \subseteq X$. The set $H = \{H_\alpha : \alpha \in \triangle\} \subseteq \tau$ is an open cover of A if $A \subseteq \bigcup_{\alpha\in\beta} H_\alpha$.*

Definition 38. *A space (X, τ) is called a compact space if and only if every open cover of X has a finite subcover.*

Remark 36.
- *$A \subseteq X$ is compact on X if and only if every open cover of A has a finite subcover.*
- *The space (X, τ) is not compact if and only if there exists open cover of X has no finite subcover.*

Example 87. *Show that $(\mathbb{R}, \tau_u)$ is not compact.*
Let $G = \{(-n, n) : n \in \mathbb{N}\}$, then G is an open cover of $\mathbb{R}$. Suppose that G has a finite subcover say $(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)$, then $\mathbb{R} = (-n_1, n_1) \cup (-n_2, n_2) \cup \ldots, (-n_k, n_k) = (-n_r, n_r)$ where $n_r = max\{n_1, n_2, \ldots, n_k\}$ which is impossible. Thus G has no finite subcover and $(\mathbb{R}, \tau_u)$ is not compact.

Example 88. *Show that $(\mathbb{R}, \tau_{cof})$ is compact, where X be an infinite set.*
Let $G = \{G_\alpha : \alpha \in \triangle\} \subseteq \tau_{cof}$, be any open cover of X. Choose $\alpha_0 \in \triangle$, $G_{\alpha_0} = X \setminus \{x_1, x_2, \ldots, x_n\}$, choose $\alpha_i \in \triangle : x_i \in G_{\alpha_i}, i = 1, \ldots, n$. Now $X = G_{\alpha_0} \cup G_{\alpha_1} \cup G_{\alpha_2} \ldots G_{\alpha_n}$. So, every open cover of X has a finite subcover.

Example 89. *Let (X, τ) be a topological space, prove that any finite subset of X is compact.*

Proof. Let $A \subset X$, and A is a finite set. Let $G = \{G_\alpha : \alpha \in \triangle\} \subseteq \tau$ be any open cover of $A = \{x_1, x_2, \ldots, x_n\}$, i.e., $A \subseteq \bigcup_{\alpha\in\triangle} G_\alpha$. Choose $\alpha_i \in \triangle : x_i \in G_{\alpha_i}, i = 1, \ldots, n$, $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$, G has a finite subcover and hence A is compact. □

Example 90. *Show that $A = (0, 1)$ is not compact in $(\mathbb{R}, \tau_u)$.*
Let $G = \{(\frac{1}{n+1}, 1) : n \in \mathbb{N}\}$ be an open cover of A. Suppose that G has a finite subcover say $(\frac{1}{n_1+1}, 1), (\frac{1}{n_2+1}, 1), \ldots, (\frac{1}{n_k+1}, 1)$, then $A \subseteq (\frac{1}{n_1+1}, 1) \cup (\frac{1}{n_2+1}, 1) \cup \cdots \cup (\frac{1}{n_k+1}, 1) = (\frac{1}{n_0+1}, 1)$ where $n_0 = max\{n_1, n_2, \ldots, n_k\}$ which is impossible, thus G has no finite subcover and hence $(0, 1)$ is not compact.

Example 91.
1. *Show that $(\mathbb{R}, \tau_\ell)$ is not compact.*
   Let $G = \{(-\infty, n) : n \in \mathbb{N}\}$, then G is an open cover of $\mathbb{R}$. Suppose that G has a finite subcover say $(-\infty, n_1), (-\infty, n_2), \ldots, (-\infty, n_k)$, then $\mathbb{R} = (-\infty, n_1) \cup (-\infty, n_2) \cup \ldots, (-\infty, n_k) = (-\infty, n_r)$ where $n_r = max\{n_1, n_2, \ldots, n_k\}$ which is impossible. Thus G has no finite subcover and $(\mathbb{R}, \tau_\ell)$ is not compact.

2. *The subspace $A = \{x : x < 0\}$ is not compact.*
   Let $G = \{(-\infty, \frac{-1}{n}) : n \in \mathbb{N}\}$ be an open cover of A. Suppose that G has a finite subcover say $(-\infty, \frac{-1}{n_1}), (-\infty, \frac{-1}{n_2}), \dots (-\infty, \frac{-1}{n_k})$, then $A \subseteq (-\infty, \frac{-1}{n_1}) \cup (-\infty, \frac{-1}{n_2}) \cup \cdots \cup (-\infty, \frac{-1}{n_k}) = (-\infty, \frac{-1}{n_0})$ where $n_0 = max\{n_1, n_2, \dots, n_k\}$ which is impossible, thus G has no finite subcover and hence $(0, 1)$ is not compact.

3. *The subspace $A = \{x : x \leq 0\}$ is compact.*
   Let $G = \{G_\alpha : \alpha \in \triangle\} \subseteq \tau$ be any open cover of A since $0 \in (-\infty, 0], \exists \alpha \in \triangle : 0 \in G_\alpha = (-\infty, r)$, where $r > 0$ it is clear that $A \subseteq (-\infty, r)$, G has a finite subcover and hence A is compact.

Remark 37. *Suppose that A and B are compact subspace of a space X, then $A \cap B$ is not compact. Consider the space $(\mathbb{R}, \tau_\ell)$ and $A = (-\infty, 0) \cup \{2\}$, $B = (-\infty, 0) \cap \{3\}$ in the same way as in example 75 (3), A and B are compact where $A \cap B = (-\infty, 0)$ is not compact be example 75 (2).*

Obvious' is the most dangerous word in mathematics. E. T. Bell